Statistical Study of In-Band Crosstalk Noise Using the Multicanonical Monte Carlo Method

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***Abstract*—In-band crosstalk can pose important limitations in an all-optical wavelength-division-multiplexed network. In this letter, the multicanonical Monte Carlo (MCMC) method is applied for the study of the statistical behavior of the in-band crosstalk noise. The proposed method is accurate, efficient, and easy to use. The obtained error probabilities are compared with a previously proposed semianalytical model and are found to be in excellent agreement. The MCMC method is also used to study the asymptotic behavior of in-band crosstalk noise.**

***Index Terms*—Crosstalk, error analysis, optical receivers, wavelength-division multiplexing (WDM).**

I. Introduction

THE PERFORMANCE of wavelength-division-multiplexing (WDM) networks can be degraded by the presence of in-band crosstalk noise. This noise arises at optical cross-connects because, due to their imperfect filtering characteristics, a small delayed version of the signal or a small portion of light from other channels at the same frequency (in a network with wavelength reuse) is routed along the same path as the signal. Since in-band crosstalk noise is at the same wavelength as the signal, it cannot be removed using additional filtering and can degrade the error probability (EP) at the receiver.

In [1], a semianalytical model for the calculation of the EP in the presence of in-band crosstalk at the receiver was proposed assuming that the pulse variations of the signal and the crosstalk components are the same. Although the semianalytic model provides a useful physical insight in the statistical nature of in-band crosstalk noise, it relies on complex numerical integration techniques and cannot be used accurately if the number of interferers exceeds 70 [1]. Also, the semianalytic model cannot be applied in situations where the crosstalk components and/or the signal partially follow different optical paths within the network and as a result, their pulse variations may vary due to dispersion, fiber nonlinearity, and the filtering characteristics of the multiplexers and demultiplexers along these paths.

In this letter, the multicanonical Monte Carlo (MCMC) [4], [6] method is applied in the study of the statistical behavior of in-band crosstalk noise. This method was first suggested as a means of computing general-probability distribution functions in communications theory in [5], where it was specifically applied to the probability density function (pdf) of polarization-mode dispersion in an optical fiber and was then used in an EP calculation in [4]. Unlike standard Monte Carlo sampling, the MCMC method increases the occurrences of the samples in the tail regions of the pdf. This allows the accurate estimation of the EP, even if its value is of the order of 10^{-10}, without an excessive number of iterations. It may also be possible to further reduce the computation time required using the random walk procedure of [7]. Another advantage of this technique is its simplicity and the fact that the signal and crosstalk channels can have different pulse variations. The results obtained with the MCMC method are shown to agree very well with the semianalytical model. The MCMC method is also used to study the asymptotic behavior of the in-band crosstalk noise as the number of interferers becomes large. As shown in [2], the decision variable at the receiver follows a chi-square distribution. The chi-square distribution is shown to provide a reasonably accurate estimate for the pdf when the number of interferers exceeds 70.

II. Application of the MCMC Method

The optical field at the input of the receiver photodiode can be represented in complex notation as $x(t) = \mathrm{Re}\{A(t)\exp(j2\pi f_0 t)\}$, where f_0 the optical frequency, $A(t)$ is the envelope of the optical field given by

$$A(t) = \sum_{m=0}^{M} c_m g_m(t) e^{j\phi_m} \tag{1}$$

where $g_m(t)$ are the pulse variations of the signal $(m = 0)$ and the interfering components $(m \geq 1)$. Assuming that $g_m(t)$ are normalized so that $(1/2)\int_{\mathrm{T}} |g_m(t)|^2 dt = 1$, c_0^2 is the number of photons in the signal, while c_m^2 is the number of photons of interferer m (for $m \geq 1$). The phases ϕ_m are due to the phase noise of the LASER sources and are assumed mutually independent and uniformly distributed inside $[0, 2\pi]$. Assuming a simple integrate-and-dump electrical filter at the receiver, the decision variable X is given by [1]

$$X = \frac{1}{2}\int_0^T |A(t)|^2\, dt = \sum_{k=0}^{M}\sum_{n=0}^{M} D_{kn} e^{j(\phi_k - \phi_n)}. \tag{2}$$

In (2), the factor $1/2$ in front of the integral of $|A(t)|^2$ is due to the complex notation adopted for $x(t)$. The coefficients D_{kn} are given by $(1/2)c_k c_n \int_{\mathrm{T}} g_k(t) g_n^*(t) dt$. If the signal and the interfering channel have the same pulse variations, $g_m(t) = g(t)$ and $D_{kn} = c_k c_n$. In order to calculate the pdf $f_X(x)$ of X, the interval S in which X takes its values is divided into small subintervals, $S_0, \ldots, S_N$, and a histogram H_k is used to measure the occurrences of X that fall inside S_k. For simplicity, the length of each interval S_k can be taken constant and equal to

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Δx. On each iteration i of the MCMC method, the estimated pdf of X is stored in the variables P_k^i and as the number of iterations increases, $P_k^i \to P_k$ where $P_k = P(X \in S_k)$.

The random samples of X are generated using the Metropolis algorithm [3]. The first sample of X is calculated by randomly selecting the phases ϕ_m and using (2). Let S_a be the subinterval in which the sample belongs. Using a symmetric zero-mean distribution, the perturbations $\Delta\phi_m$ are randomly and independently selected so that their variance is constant throughout each iteration. The phases $\phi_m + \Delta\phi_m$ are used to compute a new trial value X_{new} of X. Let S_b be the subinterval in which X_{new} belongs. This new value is accepted with a probability $p = \min(P_a^i/P_b^i, 1)$.

In each iteration, the occurrences of X inside each interval S_k are recorded in the histograms H_k^i. At the end of each iteration, the values the P_k^i are updated according to the values of the $H_k^i s$ using the recurrence relations introduced in [6]

$$P_{k+1}^{i+1} = \frac{P_{k+1}^i P_k^{i+1}}{P_k^i} \left(\frac{H_{k+1}^i}{H_k^i} \right)^{g_k^i} \tag{3}$$

where the exponents g_k^j are given by

$$g_k^i = \frac{f_k^i}{\sum_{l=1}^{i} f_k^l}, \quad f_k^l = \frac{H_k^l H_{k+1}^l}{H_k^l + H_{k+1}^l}. \tag{4}$$

It should be noted that $g_k^i = 0$, if $f_k^i = 0$. Also, $f_k^l = 0$, if $H_k^l + H_{k+1}^l = 0$. The P_k^i are normalized so that their sum, with respect to k, is equal to unity. For $i = 1$, the values of $P_k^i = P_k^1$ are all set equal to $1/N$, which means that the first iteration corresponds to standard Monte Carlo sampling. As i increases, the information gained for the pdf of X through the P_k^i is used to bias the samples and increase the occurrence of the values of X at the tails of its pdf. After the final iteration $i = Q$, the values of P_k^Q provide an estimate for $f_X(x)$ and are normalized so that $\Delta x \Sigma_{m\geq 1} P_k^Q = 1$. The values of the pdf, obtained by the MCMC method can be used in order to calculate the error probabilities P_{e1}, P_{e0} when the signal bit is "1" and "0," respectively. The value of P_{e1} is estimated using

$$P_{e1} = P(X \leq a) \cong \Delta x \sum_{k=1}^{r} P_k^Q \tag{5}$$

where r is the index of the subinterval S_r in which a belongs. A similar expression is used for the estimation of P_{e0}.

III. Results of the MCMC Method

In Fig. 1(a), the values of P_{e1} are plotted for the case of $M = 10$ interfering channels with equal amplitudes assuming that the pulse variations are the same for the signal and the interferers [i.e., $g_m(t) = g(t)$]. All the interferers have the same amplitude ($c_m = c_1$, for $m \geq 1$). The energy of the signal is taken to be $c_0^2 = 100$ photoelectrons and the amplitude of the interferers is chosen so that the optical signal-to-crosstalk ratio $\text{SXR} = c_0^2/\sigma^2$ is 20 dB. The dots correspond to the values of P_{e1} obtained using the MCMC, for $Q = 10$ iterations of 10^5 samples each. The solid line corresponds to the values obtained

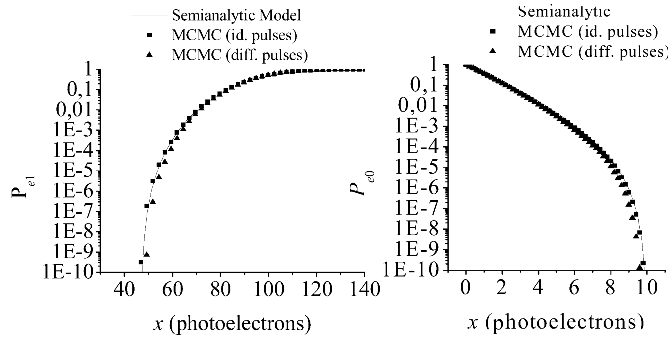

Fig. 1. Values of (a) P_{e1} and (b) P_{e0} obtained using the semianalytical model (solid-lines) and the MCMC method for identical (rectangles) and differing pulses (triangles).

using the semianalytical model. This model takes into account the crosstalk–crosstalk beating noise and computes the moment-generating function (MGF) $M_X(s)$ of the decision variable at the receiver. The computation is accomplished through the two-dimensional integral formulation of the MGF $M_X(s)$ according to which, $M_X(s)$ can be written as [1]

$$M_X(s) = \int_0^{+\infty} \rho e^{s\rho^2} \int_0^{+\infty} y M_R(y) J_0(y\rho) d\rho dy \tag{6}$$

where the $M_R(jy)$ is a product of Bessel functions J_0 of zeroth order

$$M_R(jy) = \prod_{m\geq 0} J_0(c_m y). \tag{7}$$

In [1], accurate numerical techniques for the estimation of the double integral of $M_X(s)$ were presented. The calculation of the EP is carried out using the saddle-point approximation. When the signal bit b_s is 0 (in which case $c_0 = 0$, assuming a perfect extinction ratio), the value P_{e0} of the EP is approximately

$$P_{e0} = P(X \geq a) \cong \frac{\exp\left(W(s_0)\right)}{\sqrt{2\pi W''(s_0)}} \tag{8}$$

where a is the receiver threshold and s_0 the saddle point in the positive real axis of the function $W(s)$ defined by $W(s) = \ln M_X(s) - as + \ln|s|$. Similarly, the value P_{e1} of the EP if $b_s = 1$ (in which case, $c_0 \neq 0$) is found by (8) if s_0 is replaced with the saddle point s_1 of $W(s)$ in the negative real axis. Using (8), the EP can be directly computed from the MGF of X. As shown in the figure, the two methods agree very well even for very small error probabilities (of the order of 10^{-10}). In Fig. 1(b), the values of P_{e0} obtained by the MCMC method and the semianalytical model are compared assuming that $c_0 = 0$ (implying a perfect extinction ratio). The agreement between the values obtained by the two methods is excellent in this case as well. This suggests that the double integral formulation of the MGF provides an accurate description for the statistical behavior of the decision variable at the receiver and that the numerical methods used for the calculation of the double integral (6) in [1] do indeed provide accurate results. In the case where the pulse variations $g_m(t)$ are not the same, the MCMC method can still be used to calculate the error probabilities using the appropriate D_{kn}. If, for example, $g_m(t)$ are return-to-zero Gaussian pulses approximately contained inside one bit period, each one with standard deviation σ_m and centered at $t = t_m$ then it can easily be shown that $D_{kn} = c_k c_n B_{kn} \exp(-C_{kn})$ with $B_{kn} = (2\sigma_k\sigma_n)^{1/2}/(\sigma_k^2+\sigma_n^2)^{1/2}$ and $C_{kn} = (t_k - t_n)^2/(\sigma_k^2+\sigma_n^2)/2$. In Fig. 1(a) and (b), the values of P_{e1} and P_{e0} are also plotted in this case with triangles. The signal pulse $g_0(t)$ is assumed to be centered at $t_0 = 0$. The values of σ_m for $m > 0$ have been chosen randomly from a uniform distribution so that the full-width at half-maximum (FWHM) of the interfering pulses $g_m(t)$ $(m >)$ 0 varies $\pm 30\%$, with respect to the FWHM of $g_0(t)$. The centers of the interfering pulses t_m have been randomly chosen uniformly inside $[-\sigma_0/2, \sigma_0/2]$. As seen by the figure, the values of P_{e1} and P_{e0} are somewhat different in this case and the use semianalytical model (that assumes that the pulse shapes of the interferers are the same as the signal) results in an overestimation of the error probabilities.

IV. Asymptotic Convergence

The MCMC method can be used to study the asymptotic convergence of the pdf of X. It has been theoretically shown [2] that as the number M of interferers increases, the pdf $f_X(x)$ of X asymptotically converges to the pdf of a chi-square random variable

$$f_X(x) \to \frac{1}{\sigma^2} \exp\left(-\frac{x + c_0^2}{\sigma^2}\right) I_0\left(\frac{2c_0\sqrt{x}}{\sigma^2}\right) (M \to \infty) \quad (9)$$

where $\sigma^2 = \Sigma_{m \geq 1} c_m^2$ is the in-band crosstalk noise power. In Fig. 2(a), the pdf's $f_X(x)$ of X in the case $b_s = 1$, obtained for $M = 10$ (rectangles) and $M = 70$ (triangles) are plotted assuming $c_0^2 = 100$ photoelectrons and SXR $= 20$ dB. Also plotted with rectangles is the pdf of X when $b_s = 0$, obtained by the MCMC method for $M = 70$. The solid lines correspond to the pdfs obtained by (9) in the cases $b_s = 1$ and $b_s = 0$. As shown in the figure, there is some difference between the pdf of X and its asymptotic form, for $M = 10$. However, this difference gradually diminishes as M increases. In fact, for $M = 70$, the pdf of X is fairly close to its asymptotic chi-square form both for $b_s = 0$ and $b_s = 1$. This is further illustrated in Fig. 2(b), where P_{e0} and P_{e1} obtained by the MCMC method for $M = 70$ (triangles) and the asymptotic pdf are plotted (solid lines). It is deduced that the error probabilities are approximately equal. Hence, the chi-square pdf can be used to approximately describe the statistical behavior of the decision variable when $M > 70$.

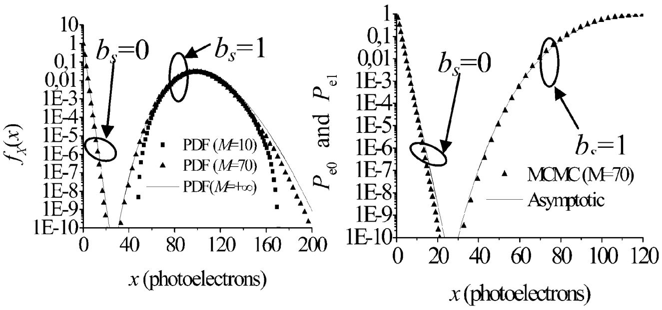

Fig. 2. (a) PDF of X obtained using the MCMC and the asymptotic chi-square form. (b) Error probabilities P_{e0} and P_{e1} obtained from the MCMC method for $M = 70$ and the asymptotic pdf.

V. Conclusion

In this letter, the MCMC method is used to study the statistical nature of the in-band crosstalk noise. The results obtained by the MCMC method are found to agree very well with the results of the previously proposed semianalytic model, which, although providing a physical insight, relies on complex numerical integration techniques. On the other hand, the MCMC method is efficient, simple to use, and can handle the case where the pulse variation of the signal and the crosstalk components are different. The asymptotic behavior of the in-band crosstalk noise is also studied using the MCMC method. It turns out that the asymptotic chi-square pdf provides an accurate estimate for the pdf of the decision variable if the number of interferers exceeds 70.

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