Single-Lifting Macaulay-Type Formulae of Generalized Unmixed Sparse Resultants

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Abstract

Resultants are defined in the toric (or sparse) context in order to exploit the structure of the polynomials as expressed by their Newton polytopes. Since determinantal formulae are not always possible, the most efficient general method for computing resultants is by rational formulae. This is made possible by Macaulay's famous determinantal formula in the dense homogeneous case, extended by D'Andrea to the toric case. However, the latter requires a lifting of the Newton polytopes, defined recursively on the dimension. Our main contribution is a single lifting function of the Newton polytopes, which avoids recursion, and yields a simpler method for computing Macaulay-type formulae of toric resultants, in the case of generalized unmixed systems, where all Newton polytopes are scaled copies of each other. In the mixed subdivision used to construct the matrices, our algorithm defines significantly fewer cells than D'Andrea's, though the formulae are same in both cases. We fully study a bivariate example and sketch how our approach extends to mixed systems of up to four polynomials, and those whose Newton polytopes have a sufficiently different face structure.

Keywords Toric resultant, Macaulay formula, Minkowski sum, mixed subdivision, generalized unmixed system

MSC classification Primary: 68W30, Secondary: 13P15, 14M25, 52B20.

1 Introduction

There are a few symbolic methods for algebraic variable elimination, including Gröbner (or standard) bases, and resultants. Both have exponential complexity in the number of variables, which is expected since the problem is NP-hard; but the latter are preferable in certain situations because they eliminate many variables at one step and can handle symbolic coefficients. Resultants also seem more efficient for solving certain classes of zero-dimensional algebraic systems. In particular, they reduce system solving to linear algebra, via matrix formulae, or to solving univariate polynomials, via the rational univariate representation of all common roots. The resultant generalizes the determinant of the coefficient matrix in the linear case, and the discriminant of a multivariate polynomial. For more information, see [CLO05, DE05, Stu02].

The toric (or sparse) resultant captures the structure of the polynomials by combinatorial means and constitutes the cornerstone of toric elimination theory [GKZ94, Stu02], [CLO05, chap.7], [DE05, chap.7]. It is an important tool in deriving new, tighter complexity bounds for system solving, Hilbert's Nullstellensatz, and related problems. These bounds depend on the polynomials' Newton polytopes and their mixed volumes, instead of total degree, which is the only parameter in classical elimination theory. In particular, if d bounds the total degree of each polynomial, the projective resultant has complexity roughly $d^{O(n)}$, whereas the toric resultant is computed in time roughly proportional to the number of integer lattice points in the Minkowski sum of the Newton polytopes.

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The resultant is defined for an overconstrained system of $n+1$ polynomials in n variables over some coefficient ring K. It is the unique, up to sign, integer polynomial over K which vanishes precisely when the system has a root in some variety X. There are two main cases:

- The projective, or classical, resultant expresses solvability of a system of dense polynomials $f_i \in K[x_1, \ldots, x_n]$ in the projective space over the algebraic closure $\overline{K}$ of K.
- The toric (or sparse) resultant expresses solvability of a system of Laurent polynomials $f_i \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ over the toric projective variety X defined by the supports of f_i, in which the torus $(\overline{K})^n$ is a dense subset.

A resultant is most efficiently expressed by a *matrix formula:* this is a generically nonsingular matrix, whose determinant is a multiple of the resultant with degree with respect to the coefficients of one polynomial equal to the corresponding degree of the resultant. For $n = 1$ there are matrix formulae named after Sylvester and Bézout, whose determinant equals the resultant. Unfortunately, such determinantal formulae do not generally exist for $n > 1$, except for specific cases, e.g. [DD01, DE03b, EM09, Khe03, KSG04, SZ94]. Macaulay's seminal result [Mac02] expresses the extraneous factor as a minor of the matrix formula, for projective resultants of (dense) homogeneous systems, thus yielding the most efficient general method for computing such resultants. There exists a method which, given a Macaulay-type formula of the resultant, constructs a determinant which equals the resultant [KK08].

Matrix formulae for the toric resultant were first constructed in [CE93]. The construction relies on a lifting of the given polynomial supports, which defines a mixed subdivision of their Minkowski sum into mixed and non-mixed cells, then applies a perturbation δ so as to define the integer points that index the matrix. The algorithm was extended in [CE00, CP93, Stu94]. In the case of dense systems, the matrix coincides with Macaulay's numerator matrix. As a corollary of this construction, one obtains a limited version of a toric effective Nullstellensatz [CE00, Sec.8].

Extending the Macaulay formula to toric resultants had been conjectured in [CE00, CLO05, Emi94, GKZ94, Stu94]; it was a major open problem in elimination theory. We cite [Stu94, p.219], where $P_{\omega,\delta}$ is the extraneous factor, and ω denotes the lifting: *"It is an important open problem to find a more explicit formula for $P_{\omega,\delta}$ in the general toric case. Does there exist such a formula in terms of some smaller resultants? This problem is closely related to the following empirical observation. For suitable choice of δ and ω, the matrix $\mathcal{M}_{\delta,\omega}$ seems to have a block structure which allows to extract the resultant from a proper submatrix. This leads to faster algorithms for computing the sparse mixed resultant."*

D'Andrea's fundamental result [D'A02] answers the conjecture by a *recursive* definition of a Macaulay-type formula, see Section 3. But this approach does not offer a global lifting, in order to address the stronger original Conjecture 1. Let M be a matrix formula, also known as Newton matrix, and $M^{(nm)}$ its submatrix indexed by points in non-mixed cells of the mixed subdivision.

Conjecture 1. [Emi94, Conj.3.1.19] [CE00, Conj.13.1] There exist perturbation vector δ and $n+1$ lifting functions for which the determinant of matrix $M^{(nm)}$ divides exactly the determinant of Newton matrix M and, hence, the toric resultant of the given polynomial system is $\det M / \det M^{(nm)}$.

Our main contribution is to give an affirmative answer to this stronger conjecture by presenting a single lifting which constructs Macaulay-type formulae for generalized unmixed systems, i.e. when all Newton polytopes are scaled copies of each other. We state our main result, to be proven in Section 4:

Theorem 2. *Algorithm B of Section 2 constructs a Macaulay-type formula for the toric resultant of an overconstrained generalized unmixed algebraic system, by means of the lifting function of Definition 6.*

Our method is generalized, in Section 6, to certain mixed systems: those with $n \leq 3$, as well as reduced systems, defined in [Zha98] to possess sufficiently different Newton polytopes. Most of these cases have been studied: reduced systems were settled in [D'A01], and bivariate systems ($n = 2$) in [DE03a], by directly establishing the extraneous factor. We expect that our approach would eventually make the single-lifting algorithm applicable to the fully general case.

Using a unique lifting function essentially means that we consider a deformed system, defined by adding a new variable t so that each input monomial x^a gets multiplied by t^b, where $b \in \mathbb{Z}$ is the lifting value of $a \in \mathbb{Z}^n$. Such deformations capture the system's behavior at toric infinity, hence lie at the heart of most theorems in toric elimination (e.g. sparse homotopies, toric resultants, the toric Nullstellensatz [Ber75, CE00, CLO05, GKZ94, HS95, Stu94]). Having a unique deformed system in defining the Macaulay-type formula might allow for further applications of this formula. Such combinatorial methods constitute one of the two main approaches for studying toric resultants, e.g. [CE00, CLO05, DD01, Min03, Stu94], the other relying on Koszul complexes and their generalizations, e.g. [DE03b, EM09, Khe03].

D'Andrea's [D'A02] recursive construction requires one to associate integer points with cells of every dimension from n to 1. Our method constructs the matrix formula directly, without recursion, by examining only n-dimensional cells. These are more numerous than the n-dimensional cells in [D'A02] but our algorithm defines significantly fewer cells totally. The disadvantage of our method is to consider extra points besides the input supports. Our single lifting algorithm is conceptually simpler and also easier to implement; see [GLW99], where the authors argue for the advantages of a single lifting over a recursive one in the context of polyhedral homotopy methods for solving algebraic systems. Existing public-domain Maple implementations cover only the original Canny-Emiris method [CE00], either standalone[1] or as part of library Multires[2].

The rest of the paper is structured as follows. The next section introduces some necessary notions, and defines the single lifting that produces Macaulay-type formulae. Section 3 recalls the recursive algorithm of [D'A02], and Section 4 proves the equivalence of the two constructions. Section 5 studies a bivariate example, and Section 6 sketches the extension of our algorithm to mixed systems.

2 Single lifting construction

This section describes our approach to defining Macaulay-formulae. For any polytopes or point sets A, B, let $\langle A \rangle$ denote the affine span (or hull) of A over $\mathbb{R}$ and $\langle A, B \rangle$ the affine span of $A \cup B$ over $\mathbb{R}$. Let $f_0, \ldots, f_n$ be polynomials with supports $A_0, \ldots, A_n \subset \mathbb{Z}^n$ and Newton polytopes

$$Q_0, \ldots, Q_n \subset \mathbb{R}^n, \; Q_i = \mathrm{CH}(A_i),$$

where $\mathrm{CH}(\cdot)$ denotes convex hull.

Our lifting shall induce a regular and fine (or tight) mixed subdivision of the Minkowski sum $\sum_{i=0}^n Q_i$ [CLO05, GKZ94]. Regularity implies the subdivision is in bijective correspondence with the face structure of the upper (or lower) hull of the Minkowski sum of $Q_0, \ldots, Q_n$ after they are lifted to $\mathbb{R}^{n+1}$. Each cell in $\mathbb{R}^n$ is written uniquely as the Minkowski sum of faces F_i of the Q_i. A fine subdivision is characterized by an equality between cell dimension and the sum of the faces' dimensions. We focus on cells of maximal dimension n, and call them maximal or, simply, cells. We distinguish them as mixed and non-mixed: the former are the Minkowski sum of n edges and a vertex. Mixed cells are i-mixed if this vertex lies in A_i. The *type* of a cell is either i-mixed or non-mixed.

Let Z be the integer lattice generated by $\sum_{i=0}^n A_i$. The Minkowski sum $\sum_{i=0}^n Q_i$ is perturbed by a vector $\delta \in \mathbb{Q}^n$, which is sufficiently small with respect to Z, and in sufficiently generic position with respect to the Q_i. The lattice points in $\mathcal{E} = Z \cap (\sum_{i=0}^n Q_i + \delta)$ are associated to a unique maximal cell of the subdivision, and this allows us to construct a matrix formula M whose rows and columns are indexed by these points. In particular, polynomial $x^{p-a_{ij}} f_i$ fills in the row indexed by the lattice point p in Definition 3.

[1]http://www.di.uoa.gr/~emiris/soft_alg.html
[2]http://www-sop.inria.fr/galaad/logiciels/multires.html

Definition 3. Let $p \in \mathcal{E}$ lie in a cell $F_0 + \cdots + F_n + \delta$ of the perturbed mixed subdivision, where F_i is a face of Q_i. The *row content (RC)* of p is (i, j), if $i \in \{0, \ldots, n\}$ is the largest integer such that F_i equals a vertex $a_{ij} \in A_i$.

Our method is based on the matrix construction algorithm of [CE00, Emi94], see also [CP93, Stu94] for generalizations. For completeness, we recall the basic steps:

1. Pick (affine) liftings $H_i : \mathbb{Z}^n \to \mathbb{R} : A_i \to \mathbb{Q}$, $i = 0, \ldots, n$.
2. Construct a regular fine mixed subdivision of the Minkowski sum $\sum_{i=0}^n Q_i$ using liftings H_i.
3. Perturb the Minkowski sum $\sum_{i=0}^n Q_i$ by a sufficiently small $\delta \in \mathbb{Q}^n$, so that integer points in $\sum_{i=0}^n Q_i + \delta$ belong to a unique cell of the subdivision, and assign *row content* to these points by Definition 3.
4. Construct resultant matrix M with rows and columns indexed by the previous integer points.

Below, we modify step 1 of this algorithm to use the lifting function of Definition 6, and shall extend the last step to produce additionally the denominator matrix. We shall refer to the modified algorithm as Alg. B.

The main idea of both our and D'Andrea's methods is that one point, say $b_{01} \in Q_0$, is lifted significantly higher. Then, the 0-summand of all maximal cells is either b_{01} or a face not containing it. In D'Andrea's case, facets not containing b_{01} correspond to different subsystems where the algorithm recurses (each time on the integer lattice specified by that subsystem). In designing a unique lifting, the issue is that points appearing in two of these subsystems may be lifted differently in different recursions. To overcome this, we introduce several points c_{ijs}, each lying in a suitable face of Q_i indexed by s, very close (with respect to Z) to every b_{ij}, which is lifted very high at recursion i by D'Andrea's method. This captures the multiple roles b_{ij} may assume in every recursion step.

Algorithm B. Our algorithm uses $\mathcal{E}$ to index the rows (and columns) of the numerator matrix of our Macaulay-type formula. We now focus on generalized unmixed systems, where

$$Q_i = k_i Q \subset \mathbb{R}^n,$$

for some n-dimensional lattice polytope Q and $k_i \in \mathbb{N}^*, i = 0, \ldots, n$. Then, the denominator shall be indexed by points lying in non-mixed cells.

Definition 4. For $i = 0, \ldots, n-2$, consider any $(n-i)$-dimensional face $F_s^{(i)} \subset Q$, where s ranges over all such faces. Take any vertex $b_{ij} \in F_s^{(i)}$, for any valid j. Let $\delta_{ijs} \in \mathbb{Q}^n$ denote a *perturbation* vector such that:

1. $b_{ij} + \delta_{ijs}$ lie in the relative interior of $k_i F_s^{(i)}$,
2. it is sufficiently small compared to lattice Z, and $\|\delta_{ijs}\| \ll \|\delta\|$, where $\| \cdot \|$ is the Euclidean norm and δ as above, and
3. it is sufficiently generic to avoid all edges in the mixed subdivision of $\sum_{i=0}^n Q_i$.

Condition 1 of Definition 4 implies that δ_{ijs} also lies in the relative interior of $k_i F_s^{(i)}$. We shall use the perturbation vectors of Definition 4 to define additional points *not* contained in the input supports.

Definition 5. We define points $c_{ijs} \in Q_i \cap \mathbb{Q}^n$, for $i = 0, \ldots, n-2$. Firstly, set $c_{011} := b_{01} + \delta_{011} \in Q_0 \cap \mathbb{Q}^n$ where δ_{011} satisfies Definition 4. Now let $\{c_{ijs} \in k_i F_s^{(i)}\}$ be the set of points defined in Q_i, where s ranges over all $(n-i)$-dimensional faces $F_s^{(i)} \subset Q$ and j over the set of indices of points in Q_i. Then, let $F_u^{(i+1)}$ be a facet of $F_s^{(i)}$ such that:

1. $k_i F_u^{(i+1)}$ does *not* contain any of the b_{ij}'s corresponding to the already defined c_{ijs}'s, and

2. $k_{i+1} F_u^{(i+1)}$ does *not* contain any of the already defined $c_{(i+1)l}$'s.

For each such facet choose a vertex $b_{(i+1)j} \in A_{i+1}$, for some j, and a suitable perturbation vector $\delta_{(i+1)ju}$ satisfying Definition 4, and set $c_{(i+1)ju} := b_{(i+1)j} + \delta_{(i+1)ju} \in Q_{i+1} \cap \mathbb{Q}^n$.

The previous definition implies a many-to-one mapping from the set of c_{ijs}'s to that of b_{ij}'s; it reduces to a bijection when restricted to a fixed face $k_i F_s^{(i)} \subset Q_i$ containing b_{ij}. Condition 1 of Definition 4 implies that c_{ijs} does not lie on a face of dimension $< n - i$ and lies in the interior of $(n-i)$-dimensional $F_s^{(i)}$. We can reduce the number of the c_{ijs}'s in Alg. B, but this would complicate the subsequent proofs.

For an application of Definition 5 when $n = 2$ see Figure 1 where Q is the unit square, and also Figure 7 where Q is a pentagon. In both examples, for illustration purposes, we define points c_{ijs} also on edges of polytope Q_1. See also Figure 2 where Q is the unit cube.

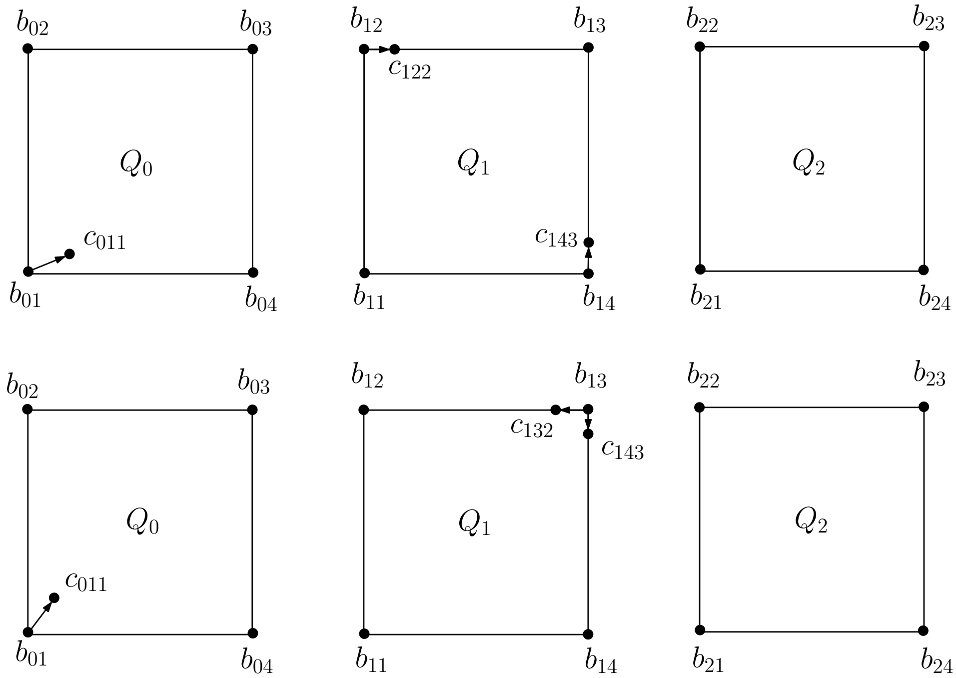

Figure 1: Two scenarios of an application of Def. 5 for 3 unit squares. Facets are numbered clockwise starting from the left vertical edge

Definition 6. Let $h_0 \gg h_1 \gg \ldots \gg h_{n-1} \gg 1$. Alg. B uses sufficiently random linear functions $H_i, i = 0, \ldots, n$, such that:

$$1 \gg H_i(a_{ij}) > 0, \text{ and } H_i \gg H_t, \; i < t,$$

where $a_{ij} \in A_i$ and $i, t = 0, \ldots, n, \; j = 1, \ldots, |A_i|$. Alg. B defines global lifting β as follows:

1. $c_{ijs} \mapsto h_i, \; c_{ijs} \in k_i F_s^{(i)} \subset Q_i, \; i = 0, \ldots, n-1$; this is called primary lifting.

2. $a_{ij} \mapsto H_i(a_{ij}), \; a_{ij} \in A_i, \; i = 0, \ldots, n$.

Let F^β denote face F lifted under β. Now c_{tjs}^β, for all valid j, s, is much higher, respectively lower, than any c_{ijs}^β, for $i > t$, respectively $i < t$. The β-induced subdivision contains edges with one or two vertices among the c_{ijs}, and edges from the Q_i. The vertex set of the upper hull of Q_i^β contains some or all of the c_{ijs}^β and the lifted vertices of Q_i.

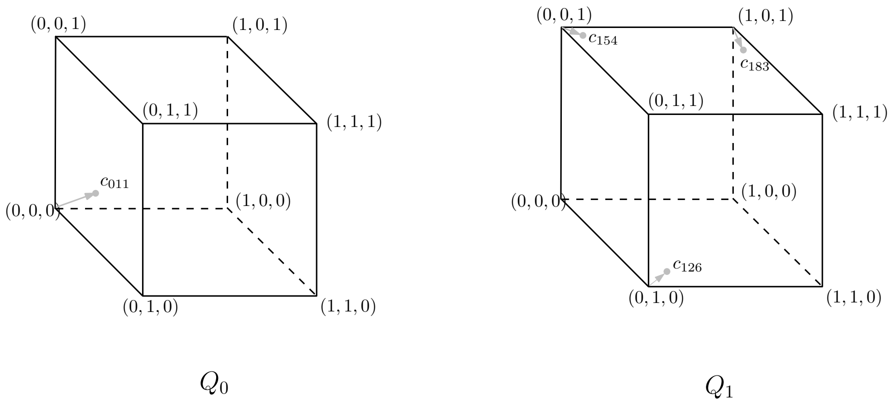

Figure 2: Application of Def. 5 when Q is the unit cube. Alg. B defines additional points only in polytopes Q_0 and Q_1

When all Q_i are simplices, as in the classical dense case, it suffices to apply a primary lifting to one point of every Q_i as in Definition 5. Thus our scheme generalizes the approach by Macaulay [Mac02].

Figure 3 shows the mixed subdivisions of three unit squares and their Minkowski sum, induced by lifting β. Here, the perturbation vectors are not sufficiently small compared to $\mathbb{Z}^2$ for illustration purposes.

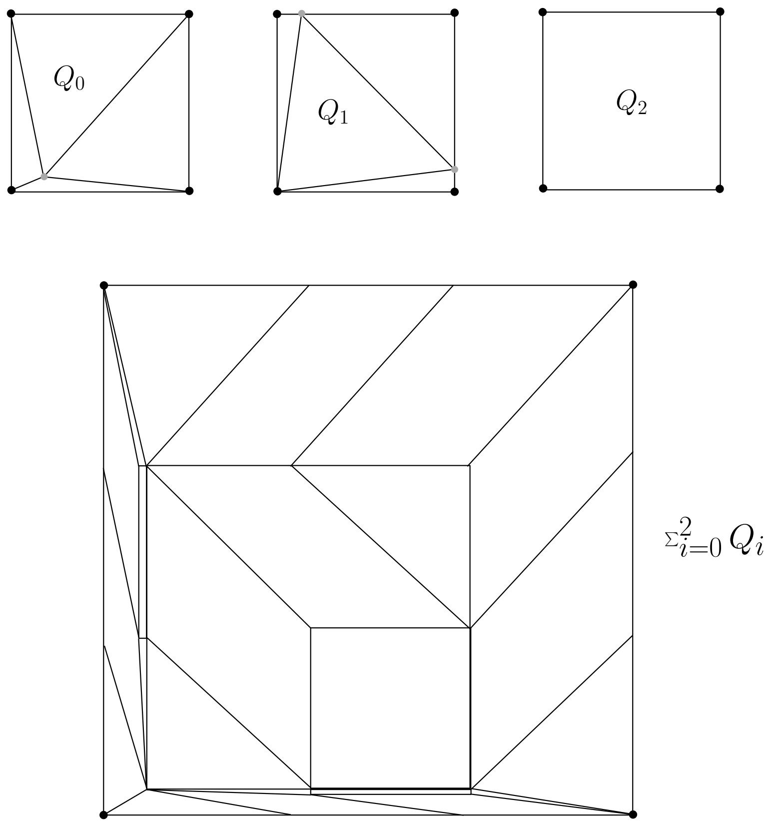

Figure 3: The mixed subdivisions of 3 unit squares and their Minkowski sum induced by lifting β

The matrix formula constructed by Alg. B is indexed by all lattice points in $\mathcal{E}$. To decide the content of each row, every point is associated to a unique (maximal) cell of the mixed subdivision according to Definition 3. The t-mixed cells contain lattice points as follows:

$$p \in k_0 E_0 + \cdots + k_{t-1} E_{t-1} + c_{tjs} + k_{t+1} E_{t+1} + \cdots + k_n E_n \cap Z,$$

for edges $E_i \subset Q$ spanning $\mathbb{R}^n$. This gives unique writing

$$p = p_0 + \cdots + p_{t-1} + (b_{tj} + \delta_{tjs}) + p_{t+1} + \cdots + p_n,\ p_i \in A_i \cap E_i.$$

Hence, the row indexed by p, as with matrix constructions in [CE00, D'A02], contains a multiple of $f_t(x)$:

$$x^{p_0+\cdots+p_{t-1}+p_{t+1}+\cdots+p_n} f_t(x),$$

and the diagonal element is the coefficient of the monomial with exponent b_{tj} in $f_t(x)$. Similarly, for the rows corresponding to lattice points in non-mixed cells.

Let us sketch the asymptotic complexity of our algorithm. Alg. B, implemented by the direct approach of [CE00], comprises of two main steps. First, the computation of the vertices of each Q_i which is typically dominated. Second, we compute RC for all $p \in \mathcal{E}$, which includes the matrix construction. Both steps can be reduced to linear programming with C constraints in V variables, and coefficient bitsize B. If we use a poly-time algorithm such as Karmarkar's [Kar84], the bit complexity is $C^{5.5}V^2B^2$, where B depends on the bitsize of the input coordinates and of δ, δ_{ijs}. It is related to the probability that the chosen perturbations are not sufficiently generic; see [CE00] for the full analysis.

Let m be the maximum number of vertices of the Q_i, r the total number of c_{ijs}'s, and let $O^*(\cdot)$ indicate that we ignore polylog factors. The linear programs have complexity $O^*(r^2B^2) = O^*(m^nB^2)$ because r is bounded by the total number $O(m^{\lfloor n/2 \rfloor})$ of faces in Q, which is quite pessimistic. In an output sensitive manner, $r = O(|\mathcal{E}|)$, because the addition of every c_{ijs} is made in order to handle at least one distinct point in $\mathcal{E}$. Hence, the complexity of constructing the Macaulay-type formula is $O^*(|\mathcal{E}|^3B^2)$. This holds for matrices in sparse and dense representation. For generalized unmixed systems, one can use $|\mathcal{E}| = O(k^n e^n D)$ from [CE00, thm.3.10], where $k = \max_i\{k_i\}$, D is the total degree of the toric resultant as a polynomial in the input coefficients, and e the basis of natural logarithms.

A better implementation finds RC for one point in a maximal cell, then enumerates all points in this cell in time proportional to their cardinality multiplied by a polynomial in m, n, B [Emi02, thm.16]. The neighbours of these points which lie outside the cell will yield new cells, so as to explore the entire Minkowski sum; detecting new cells does not increase the overall complexity. If $S \leq |\mathcal{E}|$ is the number of maximal cells containing at least one lattice point, Alg. B has complexity $O^*(Sr^2B^2 + |\mathcal{E}|) = O^*(S|\mathcal{E}|^2B^2)$, where typically, $S \ll |\mathcal{E}|$. This may be compared to the complexity of Alg. A at the end of the next section.

3 Recursive construction

This section discusses D'Andrea's recursive construction of a Macaulay-type formula [D'A02]. There are certain free parameters in the algorithm which we specify so as to obtain a version very similar to our approach.

At the input of the 0-step the algorithm may use an additional polytope mQ, for any $m \in \mathbb{R}$, which we omit by setting $m = 0$. We describe the t-th recursive step, for $t = 0, 1, \ldots, n-1$.

Algorithm A. The input are polytopes

$$l_0P^{(t)}, \ldots, l_{t-1}P^{(t)}, k_tP^{(t)}, \ldots, k_nP^{(t)} \subset \mathbb{R}^{n-t},\ l_i \in [0, k_i] \cap \mathbb{Q},$$

the integer lattice $L^{(t)}$ spanned by $\sum_{i=t}^n A_i \cap k_iP^{(t)}$, and perturbation vector $\delta_t \in \mathbb{Q}^{n-t}$. Here, $k_iP^{(t)},\ i \geq t$, is an $(n-t)$-dimensional face of k_iQ, thus $P^{(0)} = Q$. Also, $P^{(t)}$ is a facet of $P^{(t-1)}$, and $l_iP^{(t)},\ i < t$, is homothetic to $k_iP^{(t)}$. These constructions shall be specified at the Recursion Phase. Also, $L^{(0)} = Z$ and $\delta_0 = \delta$.

Construction Phase: Vertex $b_{tj} \in k_t P^{(t)} \cap A_t$ is lifted to 1. We require that $b_{tj} = c_{tjs} - \delta_{tjs}$, where s is determined by the face $k_t P^{(t)}$. All other vertices of all input polytopes are lifted to 0. This is the primary lifting which partitions the Minkowski sum of the input polytopes into a primary cell

$$l_0 P^{(t)} + \cdots + l_{t-1} P^{(t)} + b_{tj} + k_{t+1} P^{(t)} + \cdots + k_n P^{(t)} + \delta_t, \tag{1}$$

of dimension $n - t$, and several secondary cells. Each secondary cell is defined by an inner normal $v \in \mathbb{Q}^{n-t}$ to a facet of $k_t P^{(t)}$ not containing b_{tj}.

Polytopes $\sum_{i=0}^{t-1} l_i P^{(t)}, k_{t+1} P^{(t)}, \ldots, k_n P^{(t)}$ are lifted by applying the restriction of β on them. We consider β fixed throughout the algorithm. The upper hull of the Minkowski sum of the lifted polytopes induces a mixed subdivision of $\sum_{i=0}^{t-1} P^{(t)} + k_{t+1} P^{(t)} + \cdots + k_n P^{(t)}$, which is then perturbed by δ_t. The lattice points p of $L^{(t)}$ contained in the perturbed subdivision are assigned RC by Definition 3. This also assigns RC to points $p + b_{tj}$ contained in the intersection of (1) with $L^{(t)}$. Let us take care of the c_{ijs}. If point p lies in

$$(F + F_{t+1} + \cdots + F_n + \delta_t) \cap L^{(t)}, \tag{2}$$

where $F_i \subset k_i Q_i,\ i > t,\ F \subset \sum_{i=0}^{t-1} l_i P^{(t)}$, having $\mathrm{RC}(p) = (h, j)$, where $F_h = c_{hjs} = b_{hj} + \delta_{hjs}$, then the corresponding matrix row is filled in by $x^{p - b_{hj}} f_h$.

Face $F \subset \sum_{i=0}^{t-1} P^{(t)}$ in (2), can be written as $F = l_0 F_0 + \cdots + l_{t-1} F_{t-1}$, where $F_i \subset P^{(t)}$ for $i < t$. Moreover, every cell in (1) is the Minkowski sum of b_{tj} and the cell in (2).

Mixed cells of type 0 are defined here as in Section 2. A t-mixed cell with respect to Alg. A, for $t > 0$, shall have $n - t$ linear summands from polytopes $k_{t+1} P^{(t)}, \ldots, k_n P^{(t)}$ and a zero-dimensional summand from polytope $\sum_{i=0}^{t-1} l_i P^{(t)}$. This summand can be written as $l_0 p_0 + \cdots + l_{t-1} p_{t-1}$, where $p_i \in P^{(t)}$, for $i = 0, \ldots, t-1$ and $l_i p_i$ stands for a scalar multiple of p_i, seen as a vector. This leads to:

Lemma 7. *The maximal cells at step t of Alg. A are, for some j and $l_i \in [0, k_i]$, of the form:*

$$l_0 F_0 + \cdots + l_{t-1} F_{t-1} + b_{tj} + k_{t+1} F_{t+1} + \cdots + k_n F_n + \delta_t, \tag{3}$$

where F_i is the projection of a face of the upper hull of $P^{(t)}$ lifted by β, and

$$\dim(\langle F_0, \ldots, F_{t-1}, F_{t+1}, F_n \rangle) = n - t.$$

Specifically, the t-mixed cells in Alg. A are:

$$l_0 p_0 + \cdots + l_{t-1} p_{t-1} + b_{tj} + k_{t+1} E_{t+1} + \cdots + k_n E_n + \delta_t, \tag{4}$$

where $E_{t+1}, \ldots, E_n$ are projections of edges on the upper hull of $P^{(t)}$ lifted by β, $\dim(\langle E_{t+1}, \ldots, E_n \rangle) = n - t$, and points $p_i \in P^{(t)}$, for $i = 0, \ldots, t-1$.

Example 8. Consider the three pentagons of Example 22. In the 0 step of the recursion, b_{01} is lifted to 1, while all other vertices of all polygons are lifted to 0. Then, the primary cell is subdivided using lifting β. The primary and secondary cells are shown in Figure 4, left, in white and grey color respectively (also in Figure 7). To illustrate Lemma 7, consider cells 1,2 and 3 of the primary cell. They can be written as

Cell 1: $b_{01} + \mathrm{CH}(c_{122}, c_{143}, c_{154}) + b_{21}$, non-mixed.

Cell 2: $b_{01} + (c_{122}, c_{154}) + (b_{21}, b_{21})$, 1-mixed.

Cell 3: $b_{01} + \mathrm{CH}(c_{122}, b_{11}, c_{154}) + b_{21}$, non-mixed.

Now, consider the recursion step of Alg. A at the secondary cell of step 0 with respect to vector $(1, 0)$ shown in Figure 4, right. In this cell the algorithm recurses on a segment containing points $(0, 4), (0, 5), (0, 6), (0, 7)$. This segment is partitioned into new primary and secondary cells and the new primary cell is subdivided again using β. The cells are:

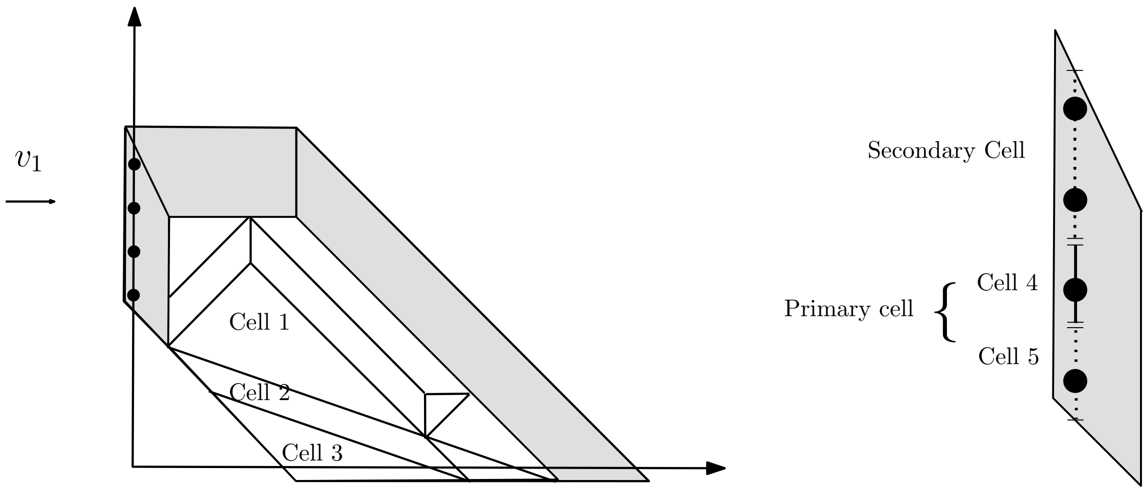

Figure 4: Example 8: 0-step (left) and 1-step of the recursion on secondary cell w.r.t. v_1 (right) of Alg. A

Secondary cell: $\frac{29}{30}b_{03} + (b_{12}, b_{13}) + b_{23}$, 2-mixed.

Cell 4: $\frac{29}{30}(b_{02}, b_{03}) + b_{11} + b_{22}$, non-mixed.

Cell 5: $\frac{29}{30}b_{02} + b_{11} + (b_{22}, b_{23})$, 1-mixed.

For details see Example 22.

Recursion Phase: When $t = n - 1$, the algorithm terminates, since it has reached the Sylvester case. Otherwise, it recurses: let $P^{(t+1)}$ be the facet of $P^{(t)}$ supported by v. The (perturbed) secondary cell corresponding to v is

$$\begin{aligned}\mathcal{F}_v = l_0 P^{(t+1)} + \cdots + l_{t-1}P^{(t+1)} + \mathrm{CH}(b_{tj}, k_t P^{(t+1)}) \\ +k_{t+1}P^{(t+1)} + \cdots + k_n P^{(t+1)} + \delta_t.\end{aligned} \tag{5}$$

Its associated diameter is

$$d_v = b_{tj} \cdot v - \min_{p \in \mathrm{CH}(b_{tj}, k_t F)} \{p \cdot v\} \in \mathbb{N}^*,$$

where $\cdot$ stands for inner product. We define two sublattices of $L^{(t)}$: $L_+^{(t)}$ is spanned by $\sum_{i=t+1}^{n} A_i \cap k_i P^{(t+1)}$ and $L_v^{(t)}$ is the sublattice orthogonal to v. They have the same dimension, so we define the (finite) index $\mathrm{ind}_v = [L_v^{(t)} : L_+^{(t)}]$, equal to the quotient of the volumes of their base cells. Let q range over the ind_v coset representatives for $L_+^{(t)}$ in $L_v^{(t)}$.

Let $l_t \in [0, k_t]$ take d_v distinct values corresponding to different values of $p \cdot v$ for all $p \in (\mathrm{CH}(b_{tj}, k_t P^{(t+1)}) + \delta_t) \cap L^{(t)}$. Note that $l_t P^{(t+1)}$ is homothetic to $k_t P^{(t+1)}$. Let $\delta_t' \in \mathbb{Q}^{n-t}$ be a translation vector such that $l_t P^{(t+1)} + \delta_t'$ contains at least one point in $(\mathrm{CH}(b_{tj}, k_t P^{(t+1)}) + \delta_t) \cap L^{(t)}$.

In particular, $l_t P^{(t+1)} + \delta_t'$ equals $k_t P^{(t+1)}$ if and only if $l_t = k_t$, and vertex b_{tj} if and only if $l_t = 0$, otherwise it equals $(\mathrm{CH}(b_{tj}, k_t P^{(t+1)}) + \delta_t) \cap H$, where H is a hyperplane parallel to a supporting hyperplane of $k_t P^{(t+1)}$; see [D'A02, lem.3.3]. By abuse of notation, in the rest of this paper we shall denote H, and the supporting hyperplanes of faces $k_t P^{(t+1)}$ and b_{tj} of the previous convex hull, as $\langle l_t P^{(t+1)} \rangle$.

Points in $(\mathcal{F}_v + \delta_t) \cap L^{(t)}$ are partitioned into d_v subsets (one per value of l_t), called *slices*, of the form

$$l_0 P^{(t+1)} + \cdots + l_{t-1}P^{(t+1)} + (l_t P^{(t+1)} + \delta_t') + k_{t+1}P^{(t+1)} + \cdots + k_n P^{(t+1)} + \delta_t \cap L^{(t)}, \tag{6}$$

which can be rearranged as

$$l_0P^{(t+1)} + \cdots + l_tP^{(t+1)} + k_{t+1}P^{(t+1)} + \cdots + k_nP^{(t+1)} + \delta_\lambda \cap L^{(t)}, \tag{7}$$

where $\delta_\lambda = \delta_t + \delta'_t$. Moreover, δ_λ can be decomposed as $\delta_\lambda^v + \delta_{\lambda v}$, where $\delta_\lambda^v \in \mathbb{Q}v$ and $\delta_{\lambda v} \in L_+^{(t)} \otimes \mathbb{Q}$. Now, every point in (7) corresponds to a point in

$$l_0P^{(t+1)} + \cdots + l_tP^{(t+1)} + k_{t+1}P^{(t+1)} + \cdots + k_nP^{(t+1)} + \delta_{\lambda v} \cap (q + L_+^{(t)}),$$

for some coset representative q. Set $\delta_{t+1} := \delta_{\lambda v} - q$, $L^{(t+1)} := L_+^{(t)}$, and observe that point p belongs to (7) if and only if point

$$p' := p - \delta_\lambda^v - q \tag{8}$$

belongs to

$$l_0P^{(t+1)} + \cdots + l_tP^{(t+1)} + k_{t+1}P^{(t+1)} + \cdots + k_nP^{(t+1)} + \delta_{t+1} \cap L^{(t+1)}. \tag{9}$$

We call this set a *piece*; δ_{t+1} carries the information to define the piece from the input polytopes and $L^{(t+1)}$. The algorithm recurses on each of the ind_v such pieces. The set

$$l_0P^{(t+1)}, \ldots, l_tP^{(t+1)}, k_{t+1}P^{(t+1)}, \ldots, k_nP^{(t+1)},\ \delta_{t+1}$$

over $L^{(t+1)}$ is exactly like the original input, only one dimension lower. This completes the algorithm.

Remark 9. Since every point p' in a piece corresponds bijectively to a point p in a slice via the monomial bijection (8), we shall often consider a piece as a subset of a slice and omit the translation.

At the end of the recursion, RC is defined on $\mathcal{E}$. Alg. A defines a partition of $\mathcal{E}$ in the form of a collection of mixed subdivisions of primary cells (of decreasing dimension). The edges of the cells of this partition, coming from polytope Q_i, are defined by any point in A_i or among the c_{ijs}, for all valid j, s, and may be multiplied by a rational number in $(0, k_i]$.

D'Andrea's algorithm uses at every construction step the matrix construction algorithm of [CE00], so its complexity is dominated by $O(|\mathcal{E}|n)$ linear programs, since every $p \in \mathcal{E}$ may require $O(n)$ of them for its image under RC to be determined. Each linear program has bit complexity $O(n^{7.5}m^2B^2)$, by Karmarkar's algorithm, where m is the maximum number of vertices of the Q_i, and B is the bitsize of the input coordinates. This process essentially decides in which slice of which secondary cell lies p. Although this subdivision contains much more cells than Alg. B, the asymptotic analysis indicates that the latter is competitive for large n; see the end of section 2 for comparing with Alg. A.

4 Equivalence of constructions

This section demonstrates that both approaches define the same Macaulay-formula. Intuitively, the single-lifting algorithm (Alg. B) has an overall effect very similar to that of Alg. A, since they both use β. The former partitions $\mathcal{E}$ into sets of points in n-dimensional cells and assigns RC, whereas Alg. A partitions $\mathcal{E}$ into subsets which, at step t, lie on the intersection of a $(n-t)$-dimensional hyperplane with an n-dimensional cell of β. Note that the intersection itself, as a subset of $\mathbb{R}^{n-t}$, does not coincide with the cell of Alg. A. However, their set difference is of infinitesimal volume and thus contains no lattice points. Although both algorithms use β to subdivide their input polytopes, they do so in a distinct fashion; Alg. B applies β to every Q_i, whereas Alg. A does so recursively to a different set of polytopes at every step.

In the rest of the paper, for simplicity, we shall omit the translation vectors δ_t. Moreover, unless otherwise stated, we shall treat every slice and piece as a polytope and not as the set of points in the intersection of this polytope with an appropriate lattice. In particular, we shall be interested only on the form of a slice or piece as a Minkowski sum of polytopes. The existence of a translation vector, for this polytope to contain integer points in the considered lattice, shall be implied.

We now establish the correspondence between the two algorithms for $t = 0$, then generalize to $t > 0$. We introduce the notation $pr.cell_i^{(X)}$, $sec.cell_i^{(X)}$, where i indicates the recursion step of Alg. A and $X \in \{A, B\}$ indicates the algorithm under consideration. At step 0 of Alg. A, b_{01} is lifted to 1, while every other vertex of all input polytopes to 0; this creates a primary cell

$$pr.cell_0^{(A)} := b_{01} + k_1 Q + \cdots + k_n Q,$$

and several secondary cells of the form

$$sec.cell_0^{(A)} := \mathrm{CH}(b_{01}, k_0 P^{(1)}) + k_1 P^{(1)} + \cdots + k_n P^{(1)},$$

each corresponding to a facet $P^{(1)}$ of Q not containing b_{01}. In Alg. B, c_{011} plays the role of b_{01} and this leads to a group of cells covering the corresponding primary cell

$$pr.cell_0^{(B)} := c_{011} + k_1 Q + \cdots + k_n Q,$$

and several groups of cells, each group covering

$$sec.cell_0^{(B)} := \mathrm{CH}(c_{011}, k_0 P^{(1)}) + k_1 P^{(1)} + \cdots + k_n P^{(1)},$$

which is a typical n-dimensional secondary cell with respect to Alg. B. Not all cells in $sec.cell_0^{(B)}$ may have $k_i P^{(1)}$ as a summand. Those who do not, have a summand where some or all of the vertices of $k_i P^{(1)}$ are replaced by the corresponding additional points c_{ijs} from Definition 5.

Remark 10. All cells within $pr.cell_0^{(A)}$ and $pr.cell_0^{(B)}$ differ only at their first summand; the former are of the form $b_{01} + F_1 + \cdots + F_n$, whereas the latter are $c_{011} + F_1 + \cdots + F_n$, where F_i is a face of Q_i, since β is used by both algorithms to subdivide $Q_1 + \cdots + Q_n$, and $c_{011} = b_{01} + \delta_{011}$.

Lemma 11. *$pr.cell_0^{(A)} \cap \mathcal{E} = pr.cell_0^{(B)} \cap \mathcal{E}$, and points in this set are assigned the same RC under both algorithms.*

Proof. Recall that $\delta_0 = \delta$ and consider the subdivision of $\sum_{i=0}^{n} Q_i$ induced by β and compare $pr.cell_0^{(A)} + \delta$ and $c_{011} + Q_1 + \cdots + Q_n + \delta = b_{01} + \delta_{011} + Q_1 + \cdots + Q_n + \delta$. These polytopes differ by δ_{011}, which is very small. Moreover, by the choice of δ, the boundary of $pr.cell_0^{(A)} + \delta$ has no points in Z. Since, by Definition 4, $\|\delta\| \gg \|\delta_{011}\|$, the two polytopes contain the same Z-points. This settles the first claim. The second claim follows from Remark 10 and the fact that the two subdivisions may only differ in cells having vertex b_{01} instead of c_{011}. Since $c_{011} - b_{01} = \delta_{011}$ is very small compared to Z, even these cells contain the same Z-points. □

Example 12. Let us return to our running Example 22. It holds that $pr.cell_0^{(A)} \cap \mathcal{E} = pr.cell_0^{(B)} \cap \mathcal{E}$. Now, consider points $(8,1), (7,2)$ and $(4,4)$, see Figures 7,8. They belong to cells of $pr.cell_0^{(A)}$ and $pr.cell_0^{(B)}$ as in the following table:

point	cell in $pr.cell_0^{(A)}$	cell in $pr.cell_0^{(B)}$	type	RC
$(8,1)$	$b_{01} + c_{154} + \mathrm{CH}(b_{22}, b_{24}, b_{25})$	$c_{011} + c_{154} + \mathrm{CH}(b_{22}, b_{24}, b_{25})$	non-mixed	$(1,5)$
$(7,2)$	$b_{01} + (c_{143}, c_{154}) + (b_{23}, b_{24})$	$c_{011} + (c_{143}, c_{154}) + (b_{23}, b_{24})$	0-mixed	$(0,1)$
$(4,4)$	$b_{01} + (c_{143}, c_{154}) + (b_{22}, b_{23})$	$c_{011} + (c_{143}, c_{154}) + (b_{22}, b_{23})$	0-mixed	$(0,1)$

Note that, for simplicity, we have omitted the global perturbation vector δ.

Each $sec.cell_0^{(A)}$ is divided by Alg. A into slices

$$l_0 P^{(1)} + k_1 P^{(1)} + \cdots + k_n P^{(1)},$$

one for each value of $l_0 \in [0, k_0]$. Each slice is partitioned into pieces on which Alg. A recurses producing $(n-1)$-dimensional primary cell

$$pr.cell_1^{(A)} := l_0 P^{(1)} + b_{1j} + k_2 P^{(1)} + \cdots + k_n P^{(1)}, \tag{10}$$

and secondary cells

$$sec.cell_1^{(A)} := l_0 P^{(2)} + CH(b_{1j}, k_1 P^{(2)}) + k_2 P^{(2)} + \cdots + k_n P^{(2)}. \tag{11}$$

Every piece of a given slice lies on lattice $L^{(1)}$ and can be thought of as the intersection of a translation of that slice, regarded as a polytope, with $L^{(1)}$. Recall that, by Remark 9, we shall consider a piece as subset of a slice.

Similarly to Alg. A, we can partition the corresponding $sec.cell_0^{(B)}$ into slices:

$$l'_0 P^{(1)} + k_1 P^{(1)} + \cdots + k_n P^{(1)},$$

by intersecting $\mathrm{CH}(c_{011}, k_0 P^{(1)})$ with a hyperplane parallel to (a supporting hyperplane of) $k_0 P^{(1)}$. Recall that we denote this hyperplane as $\langle l'_0 P^{(1)} \rangle$.

Remark 13. Observe that each slice of $sec.cell_0^{(B)}$ (resp. $sec.cell_0^{(A)}$) parameterized by l'_0 (resp. l_0), is homothetic to a facet of this secondary cell, supported by $\langle k'_0 P^{(1)} \rangle$ (resp. $\langle k_0 P^{(1)} \rangle$). Moreover, this homothecy is defined by a homothecy only on the first summand $k_0 P^{(1)}$ of this facet.

Example 14. To illustrate Remark 13, consider in our running Example 22 the secondary cell with respect to Alg. A

$$\mathcal{F}_{v_3} = \mathrm{CH}(b_{01}, k_0 F_{v_3}) + k_1 F_{v_3} + k_2 F_{v_3} + \delta,$$

defined by the facet $F_{v_3} = ((3,0),(1,2))$ of Q supported by $v_3 = (-1,-1)$, and its slice

$$(l_0 F_{v_3} + \delta') + k_1 F_{v_3} + k_2 F_{v_3} + \delta, \tag{12}$$

where $l_0 = \frac{32}{60}$ and $\delta' = (\frac{7}{15}, 0)$. This slice contains the integer points $(11,0),(10,1),(9,2),(8,3),(7,4),$ $(6,5),(5,6),(4,7)$ and is the dashed segment in Figure 5. It is homothetic to the facet

$$k_0 F_{v_3} + k_1 F_{v_3} + k_2 F_{v_3} + \delta \tag{13}$$

of $\mathcal{F}_{v_3}$ and the homothecy is defined by the homothecy $l_0 F_{v_3} + \delta'$ of the 0-summand $k_0 F_{v_3}$ of the facet, see Figure 5. The second slice of $\mathcal{F}_{v_3}$ is

$$\left(\frac{1}{30} F_{v_3} + (\frac{29}{30}, 0)\right) + k_1 F_{v_3} + k_2 F_{v_3} + \delta \tag{14}$$

and contains integer points $(10,0),(9,1),(8,2),(7,3),(6,4),(5,5),(4,6)$. It is homothetic to the facet (13) of $\mathcal{F}_{v_3}$ and the homothecy is defined by the homothecy $\frac{1}{30} F_{v_3} + (\frac{29}{30}, 0)$ of the 0-summand $k_0 F_{v_3}$ of the facet, see Figure 5 (dotted segment).

Hyperplanes $\langle l'_0 P^{(1)} \rangle$ and $\langle l_0 P^{(1)} \rangle$ are identical; they differ only on the homothecy on $k_0 P^{(1)}$ expressed by l'_0 and l_0 respectively. Obviously, $l'_0 \approx l_0$ because $c_{011} \approx b_{01}$. Note that we omit the translation vector so that the slice lies in $sec.cell_0^{(B)}$. Thus, corresponding slices contain the same points in the lattice $L^{(0)} = Z$. This, moreover, leads to the following extension of Lemma 11.

Lemma 15. *Every maximal cell of the subdivision induced by β on $pr.cell_1^{(A)}$ corresponds to the intersection of a unique maximal cell of the same type in $sec.cell_0^{(B)}$, with a slice defined by hyperplane $\langle l'_0 P^{(1)} \rangle$, for some l'_0. The cells contain the same points in $L^{(1)}$, with the same image under RC.*

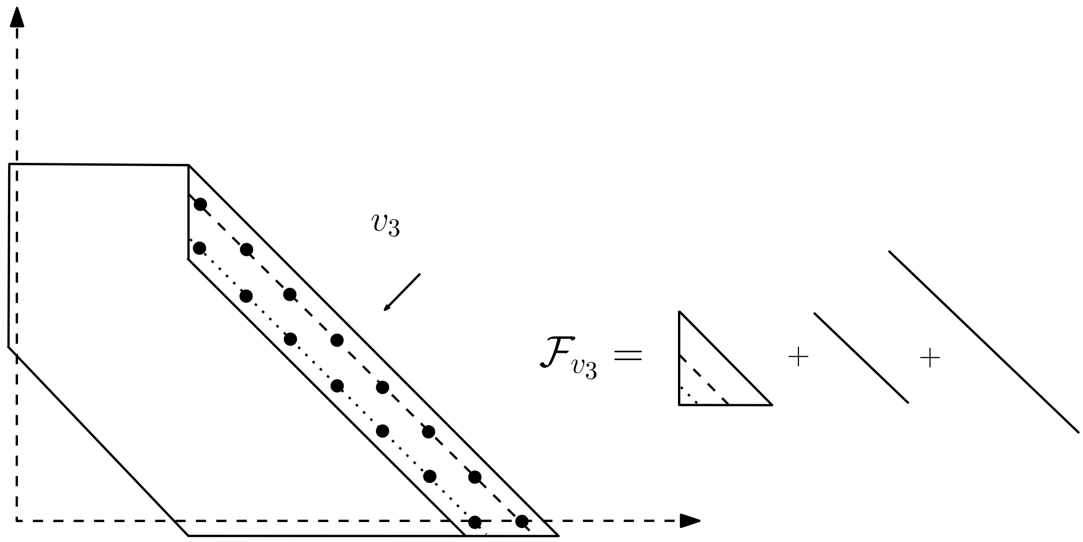

Figure 5: Example 14: The secondary cell w.r.t. $(-1,-1)$ of the 0-step of Alg. A and its two slices

Proof. Any maximal cell in $pr.cell_1^{(A)}$ has the form $l_0F_0 + b_{1j} + k_2F_2 + \cdots + k_nF_n$, where faces $F_i \subset P^{(1)}, i = 0, 2, \ldots, n$, have dimensions adding up to $n-1$. Recall $pr.cell_1^{(A)}$ lies on a slice of $sec.cell_0^{(A)}$ parameterized by the value of l_0 hence, when β is employed, it gives rise to the same subdivision in every such primary cell. By construction, subspace $\langle b_{01}, F_0\rangle$ is orthogonal and complementary to $\langle P^{(1)}\rangle$.

In $k_1P^{(1)}$, point c_{1js} is lifted sufficiently higher than any other, so there exist maximal cells in $sec.cell_0^{(B)}$ that has it as summand. The other summands are induced by β on $\mathrm{CH}(c_{011}, k_0P^{(1)})$, $k_2P^{(1)}, \ldots, k_nP^{(1)}$. These n-dimensional cells of Alg. B correspond, when intersected with the slice parameterized by $\langle l'_0P^{(1)}\rangle$, to $(n-1)$-dimensional cells in $pr.cell_1^{(A)}$. It is straightforward to show that, for $l'_0 \in [0, k_0]$ and any β-induced cell in this Minkowski sum, its intersection with the slice defined by $\langle l'_0P^{(1)}\rangle$ is a β-induced cell in $l'_0P^{(1)} + k_2P^{(1)} + \cdots + k_nP^{(1)}$

There exists $l'_0 \approx l_0$ that establishes the Lemma, because β is applied to $(n-1)$-dimensional Minkowski sums which are almost identical, and the effect of b_{1j} and c_{1js} is the same in what concerns the lattice points in corresponding cells, following the proof of Lemma 11. □

Example 16. We shall return to our running example to illustrate Lemma 15. Consider the slice

$$(l_0F_{v_3} + \delta') + k_1F_{v_3} + k_2F_{v_3} + \delta \tag{15}$$

of the secondary cell with respect to Alg. A

$$sec.cell_0^{(A)} = \mathrm{CH}(b_{01}, k_0F_{v_3}) + k_1F_{v_3} + k_2F_{v_3} + \delta,$$

where $l_0 = \frac{32}{60}$, $\delta' = (\frac{7}{15}, 0)$, $\delta = (-\frac{1}{30}, -\frac{1}{30})$, see also equation (27). This slice is obtained by intersecting $\mathrm{CH}(b_{01}, b_{04}, b_{05})$ with the hyperplane $\langle l_0F_{v_3}\rangle := \langle \frac{32}{60}F_{v_3} + (\frac{7}{15}, 0)\rangle$, and contains integer points $(11,0), (10,1), (9,2), (8,3), (7,4), (6,5), (5,6), (4,7)$ in L. The corresponding slice of $sec.cell_0^{(B)}$ is obtained by intersecting $\mathrm{CH}(c_{011}, b_{04}, b_{05})$ with the hyperplane $\langle l'_0F_{v_3}\rangle := \langle \frac{639}{1199}F_{v_3} + (\frac{1274}{2725}, \frac{28}{89925})\rangle$, see Figure 6 (dotted segment). It contains the same points in L.

Slice (15) of $sec.cell_0^{(A)}$ contains two pieces in $L^{(1)} := L_+ = \langle (9,0), (7,2)\rangle \cong 2\mathbb{Z}$:

$$piece_0 := \frac{32}{60}F_{v_3} + k_1F_{v_3} + k_2F_{v_3} + (-\frac{17}{30}, -\frac{31}{30}), \tag{16}$$

$$piece_1 := \frac{32}{60}F_{v_3} + k_1F_{v_3} + k_2F_{v_3} + (\frac{13}{30}, -\frac{61}{30}). \tag{17}$$

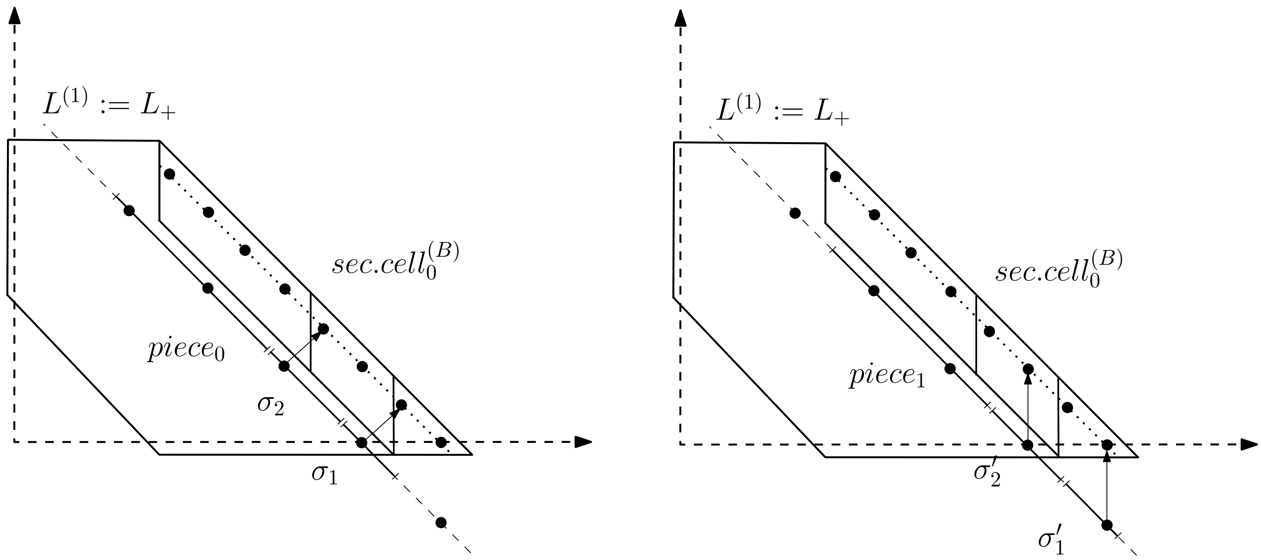

Figure 6: Example 16: The two pieces of the secondary cell w.r.t. $(-1,-1)$ of Alg. A and the correspondence between their cells and the cells of the similar secondary cell w.r.t. Alg. B

Piece (16) is partitioned into a primary cell $\frac{32}{60}F_{v_3} + b_{15} + k_2F_{v_3} + (-\frac{17}{30}, -\frac{31}{30})$ and a secondary cell $\frac{32}{60}b_{04} + k_1F_{v_3} + b_{24} + (-\frac{17}{30}, -\frac{31}{30})$. Then, lifting β induces a mixed subdivision on the primary cell consisting of the cells

$$\sigma_1 = \frac{32}{60}F_{v_3} + b_{15} + b_{25} + (-\frac{17}{30}, -\frac{31}{30}) \text{ and } \sigma_2 = \frac{32}{60}b_{04} + b_{15} + k_2F_{v_3} + (-\frac{17}{30}, -\frac{31}{30}).$$

Cell σ_1 is non-mixed and contains point $(9,0) \in L_+$, which translates to point $(10,1) \in L$. This cell corresponds to the intersection of the slice of $sec.cell_0^{(B)}$, defined by hyperplane $\langle l_0' F_{v_3} \rangle$, with its non-mixed cell $\mathrm{CH}(c_{011}, b_{04}, b_{05}) + c_{15} + b_{25} + \delta$. Cell σ_2 is 1-mixed and contains the point $(7,2) \in L_+$ which translates to the point $(8,3) \in L$. This cell corresponds to the intersection of the slice of $sec.cell_0^{(B)}$, defined by hyperplane $\langle l_0' F_{v_3} \rangle$, with the 1-mixed cell with respect to Alg. B $(c_{011}, b_{04}) + c_{154} + (b_{24} + b_{25}) + \delta$, see Figure 6,(left).

The second piece (17) is partitioned into a primary cell $\frac{32}{60}F_{v_3} + b_{15} + k_2F_{v_3} + (\frac{13}{60}, -\frac{61}{30})$ and a secondary cell $\frac{32}{60}b_{04} + k_1F_{v_3} + b_{24} + (\frac{13}{60}, -\frac{61}{30})$. Lifting β induces a mixed subdivision on the primary cell consisting of the cells

$$\sigma_1' = \frac{32}{60}F_{v_3} + b_{15} + b_{25} + (\frac{13}{60}, -\frac{61}{30}) \text{ and } \sigma_2' = \frac{32}{60}b_{04} + b_{15} + k_2F_{v_3} + (\frac{13}{60}, -\frac{61}{30}).$$

The former is non-mixed and contains point $(11,-2) \in L_+$ corresponding to $(11,0) \in L$. It corresponds to the intersection of the slice cell of $sec.cell_0^{(B)}$, defined by hyperplane $\langle l_0' F_{v_3} \rangle$, with its non-mixed cell $\mathrm{CH}(c_{011}, b_{04}, b_{05}) + c_{154} + b_{25} + \delta$. Cell σ_2' is 1-mixed and contains the integer point $(9,0) \in L_+$ corresponding to point $(9,2) \in L$. It corresponds to the intersection of the slice defined by hyperplane $\langle l_0' F_{v_3} \rangle$ with the 1-mixed cell of $sec.cell_0^{(B)}$ $(c_{011}, b_{04}) + c_{154} + (b_{24} + b_{25}) + \delta$, see Figure 6, (right).

In each $sec.cell_0^{(B)}$ we distinguish 2 types of cells: cells in

$$pr.cell_1^{(B)} := \mathrm{CH}(c_{011}, k_0P^{(1)}) + c_{1js} + k_2P^{(1)} + \cdots + k_nP^{(1)}, \tag{18}$$

which, by Lemma 15, contains exactly the integer points in all primary cells of Alg. A of the form (10) (for each slice/coset), and for each facet $P^{(2)}$ of $P^{(1)}$, cells in

$$sec.cell_1^{(B)} := \mathrm{CH}(c_{011}, k_0P^{(2)}) + \mathrm{CH}(c_{1js}, k_1P^{(2)}) + k_2P^{(2)} + \cdots + k_nP^{(2)}. \tag{19}$$

Note that both $pr.cell_1^{(B)}$ and $sec.cell_1^{(B)}$ are n-dimensional, whereas $pr.cell_1^{(A)}$ and $sec.cell_1^{(A)}$ are $(n-1)$-dimensional.

Remark 17. Every maximal cell in $sec.cell_1^{(B)}$ must have summands $F_0 = \mathrm{CH}(c_{011}, G_0)$, $F_1 = \mathrm{CH}(c_{1js}, G_1)$, for some $G_0 \subset k_0P^{(2)}$ and $G_1 \subset k_1P^{(2)}$.

A similar argument as in Lemma 15, implies that (19) contains exactly the integer points in the union of all secondary cells (11) defined over the various values of $l_0 \in [0, k_0]$, for a given j. The recursion steps of Alg. A, for $t \geq 2$ are defined over a chain of facets $P^{(2)} \supset P^{(3)} \supset \cdots \supset P^{(n-1)}$. Hence, every $pr.cell_t^{(A)}$, for $t > 1$, contains integer points in $sec.cell_1^{(B)} \cap Z$. Therefore, we generalize the correspondence between the two algorithms by focusing on $sec.cell_1^{(B)}$.

Lemma 18. (Main) *Every maximal cell of the subdivision induced by β on $pr.cell_t^{(A)}$, for $t \geq 2$, corresponds to the intersection of hyperplane $\langle l'_{t-1}P^{(t)}\rangle$, for some $l'_{t-1} \approx l_{t-1} \in [0, k_{t-1}] \cap \mathbb{Q}$, with a unique maximal cell in $sec.cell_1^{(B)}$, of the same type. The cells contain the same points in lattice $L^{(t)}$ with the same image under RC.*

Proof. Primary cells of step t lie on $(n-t)$-dimensional slices of the $(n-t+1)$-dimensional $sec.cell_{t-1}^{(A)}$, parameterized by the value of $l_{t-1} \in [0, k_{t-1}]$:

$$l_0P^{(t)} + \cdots + l_{t-1}P^{(t)} + k_tP^{(t)} + \cdots + k_nP^{(t)}. \tag{20}$$

Similarly to Remark 13, let $l_0, \ldots, l_{t-1},\ l_i \in [0, k_i] \cap \mathbb{Q}$, define the homothecies on the first t summands of (20) and the corresponding hyperplanes $\langle l_0P^{(t)}\rangle, \ldots, \langle l_{t-1}P^{(t)}\rangle$. Note, that $pr.cell_t^{(A)}$ is a subset of (20) and is subdivided by β into maximal cells of the form (3).

Intersecting $sec.cell_1^{(B)}$ with the above hyperplanes, yields a $(n-t)$-dimensional subset:

$$l'_0P^{(t)} + \cdots + l'_{t-1}P^{(t)} + k_tP^{(t)} + \cdots + k_nP^{(t)}. \tag{21}$$

This subset can also be obtained by directly intersecting $sec.cell_1^{(B)}$ with $\langle l_{t-1}P^{(t)}\rangle$. Now, $l'_i \approx l_i$, for $i = 0, 1, \ldots, t-1$ because $c_{ijs} \approx b_{ij}$. For $i = 0, \ldots, t-1$, each l'_i defines a hyperplane $\langle l'_iP^{(t)}\rangle$ identical to $\langle l_iP^{(t)}\rangle$, except on the homothecy on the i-th summand. Hence, (21) is very similar to (20) in the sense that they contain the same integer points in $L^{(t)}$ and their volumes differ infinitesimally.

By Definition 5 there exist n-dimensional cells in $sec.cell_1^{(B)}$ which have c_{tjs} as a summand. The intersection of each of these cells with (21) shall also have c_{tjs} as a summand, because this is the only point lifted highest in $P^{(t)}$. These cells correspond to the primary cell with respect to Alg. A of the slice (20). Moreover, this intersection is a β-induced cell in (21):

$$l'_0F_0 + \cdots + l'_{t-1}F_{t-1} + c_{tjs} + k_{t+1}F_{t+1} + \cdots + k_nF_n, \tag{22}$$

which contains the same integer points as (3). Since β is applied on $(n-t)$-dimensional polytopes which are almost identical, both (3) and (22) are of the same type. ☐

Corollary 19. *Using the notation of Lemma 7, in particular for t-mixed cells of Alg. A in the form of (4), a t-mixed cell of Alg. B is of the form:*

$$k_0E_0 + \cdots + k_{t-1}E_{t-1} + c_{tjs} + k_{t+1}E_{t+1} + \cdots + k_nE_n + \delta_t\ \cap L,$$

where E_i is the projection of an edge of Q^β,

(a) $\langle E_0, \ldots, E_{t-1}\rangle$ is a t-dimensional space complementary to $\langle P^{(t)}\rangle$, and for $i < t$, $k_iE_i = (c_{ijs}, k_ip_i)$, where $p_i \in P^{(i)}$ in Lemma 7, and

(b) edges $E_{t+1}, \ldots, E_n$ are the same as in (4) at Lemma 7.

Proof. For $t = 0$, the Corollary follows from Remark 10.

All 1-mixed cells with respect to Alg. B lie in (18), since every maximal cell in it has c_{1js} as a summand. By Lemma 15, edges $k_2E_2, \ldots, k_nE_n$ span the $(n-1)$-dimensional space $\langle P^{(1)} \rangle$. Hence, edge k_0E_0 has to be of the form (c_{011}, k_0p_0), where $p_0 \in P^{(1)}$, by Lemma 15, is as in Lemma 7,(4).

Similarly, Lemma 18 implies that for $t > 1$, the last $(n-t)$ edges of any t-mixed cell with respect to Alg. B span the $(n-t)$-dimensional space $\langle P^{(t)} \rangle$, because β induces the same subdivision on the last $n-t$ summands of (20) and (21). For the cell to be maximal, $\langle k_0E_0, \ldots, k_{t-1}E_{t-1} \rangle$ must be a t-dimensional space complementary to $\langle P^{(t)} \rangle$. By construction (see proof of Lemma 18), each k_iE_i, for $i < t$, is an edge in $\mathrm{CH}(c_{ijs}, k_iP^{(t)})$ of the form (c_{ijs}, k_ip_i), where $p_i \in P^{(t)}$ is as in Lemma 7,(4). □

We now consider non-mixed cells, by extending Corollary 19:

Corollary 20. *Consider any non-mixed cell of Alg. A, which has the form of (3) in Lemma 7. It corresponds to cell:*

$$CH(c_{011}, k_0F_0) + \cdots + CH(c_{(t-1)js}, k_{t-1}F_{t-1}) + c_{tjs} + k_{t+1}F_{t+1} + \cdots + k_nF_n,$$

which is a non-mixed cell defined by β, where

(a) the $F_0, \ldots, F_{t-1}$ are projections of faces in Q^β, for $i < t$, and

$$\langle CH(c_{011}, k_0F_0), \ldots, CH(c_{(t-1)js}, k_{t-1}F_{t-1}) \rangle$$

is a t-dimensional space complementary to $\langle F_{t+1}, \ldots, F_n \rangle$,

(b) $F_0, \ldots, F_{t-1}, F_{t+1}, \ldots, F_n$ are the same in both cells.

For an illustration of Corollaries 20, 19, see Table 1 in our running Example 22. We have shown that each row of the constructed matrices, indexed by points of $\mathcal{E}$ lying in a mixed or non-mixed cell, is identical for both algorithms, where $\mathcal{E}$ is the same pointset for both algorithms.

Theorem 21. *The Macaulay-type formula for the toric resultant of generalized unmixed systems constructed by Alg. B and that constructed by Alg. A, implementing D'Andrea's approach [D'A02], are identical.*

As a consequence of Theorem 21 and [D'A02, Thm. 3.8], follows Theorem 2.

5 A bivariate example

This section details the following example.

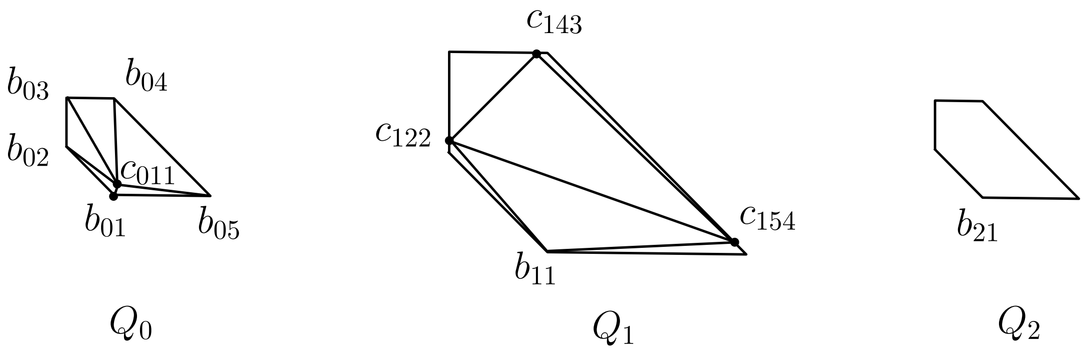

Figure 7: Input polygons of Exam. 22 and their subdivisions induced by the lifting of Def. 6

Example 22. Let $n = 2$, Q be the pentagon with vertices $\{(1,0), (0,1), (0,2), (1,2), (3,0)\}$, $k_0 = k_2 = 1$, $k_1 = 2$. The input polygons are $Q_i = k_iQ$, $i = 0,1,2$ and the input supports are $A_0 = A_2 =$

$\{(1,0),(0,1),(0,2),(1,2),\ (3,0)\}$, and $A_1 = \{(2,0),(0,2),\ (0,4),(2,4),(6,0)\}$. The lattice generated by $\sum_{i=0}^{2} A_i$ is $\mathbb{Z}^2$. The normals to the facets of Q not containing vertex $(1,0)$ are $v_1 = (-1,0)$, $v_2 = (0,-1)$, $v_3 = (-1,-1)$. Let $\delta = (-1/30,\ -1/30)$ be the global perturbation vector. See Figure 7.

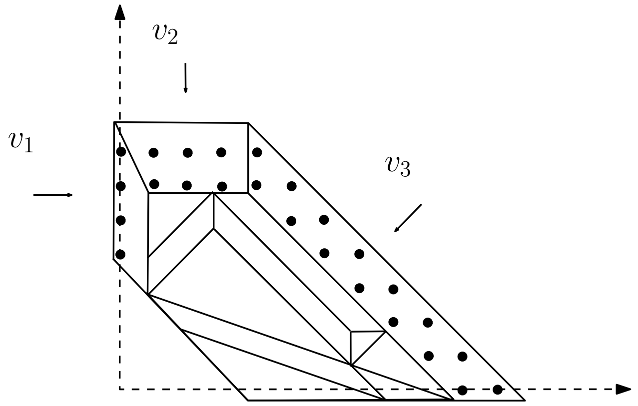

Figure 8: Exam. 22: 0-step recursion of Alg. A

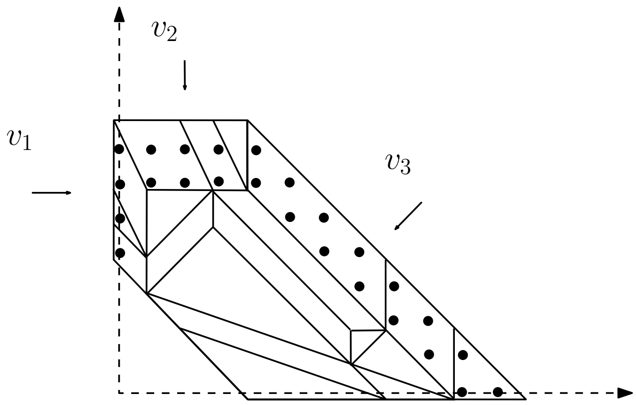

Figure 9: Exam. 22: The mixed subdivision induced by Alg. B

Alg. B: We fix vertices of the input polygons in order to define the additional points required by Definition 6. Let $b_{01} := (1,0) \in Q_0$, $b_{12} := (0,2)$, $b_{14} := (2,4)$, $b_{15} := (6,0) \in Q_1$, and perturbation vectors $\delta_{011} = (\frac{1}{1000}, \frac{1}{1500})$, $\delta_{122} = (0, \frac{1}{2000})$, $\delta_{143} = (-\frac{1}{3000}, 0)$, $\delta_{154} = (-\frac{1}{2000}, \frac{1}{2000})$. In the subdivision of $\sum_{i=0}^{2} Q_i$, consider the integer points and their cells (Figure 9):

point	cell in secondary cell w.r.t. v_2 under Alg. B	type
$(1,7),(2,7)$	$(c_{011},(0,2)) + ((0,4),c_{143}) + (0,2) + \delta$	2-mixed
$(3,7)$	$(c_{011},(0,2)) + c_{143} + ((0,2),(1,2)) + \delta$	1-mixed

where summands come from Q_0, Q_1, Q_2 respectively. These cells together with cell

$$\sigma = \mathrm{CH}(c_{011},(0,2),(1,2)) + c_{143} + (1,2) + \delta,$$

and some infinitesimal cells which do not contain any integer points, correspond to the secondary cell with respect to v_2 of Alg. A, which contains the same integer points. Points $(1,7),(2,7),\ (3,7)$ correspond (via an appropriate translation) to points of a piece of the secondary cell on which Alg. A recurses. Cell σ does not contain any integer points because of the choice of δ_{ijs}, δ.

Now, consider the points corresponding to a piece of the secondary cell with respect to v_3, of Alg. A, and their cells in the subdivision induced by β under Alg. B:

point	cell in secondary cell w.r.t. v_3 under Alg. B	type
$(4,7),(5,6),$ $(6,5),\ (7,4)$	$(c_{011},(1,2)) + (c_{154},c_{143}) + (1,2) + \delta$	2-mixed
$(8,3),(9,2)$	$(c_{011},(1,2)) + c_{154} + ((3,0),(1,2)) + \delta$	1-mixed
$(10,1),(11,0)$	$\mathrm{CH}(c_{011},(3,0),(1,2)) + c_{154} + (3,0) + \delta$	non-mixed

Consider the piece of the secondary cell with respect to v_1, of Alg. A. Points in it lie in the following cells of Alg. B:

point	cell in secondary cell w.r.t. v_1 under Alg. B	type
$(0,4)$	$(c_{011},(0,1)) + c_{122} + ((0,1),(0,2)) + \delta$	1-mixed
$(0,5)$	$\mathrm{CH}(c_{011},(0,1),(0,2)) + c_{122} + (0,2) + \delta$	non-mixed
$(0,6),(0,7)$	$(c_{011},(0,3)) + (c_{122},(0,4)) + (0,2) + \delta$	2-mixed

Alg. A: b_{01} is lifted to 1, all other vertices of all polygons are lifted to 0. This partitions $Q_0+Q_1+Q_2$ into a primary cell $b_{01}+Q_1+Q_2$ and 3 secondary cells corresponding to v_1, v_2, v_3, normals to the facets of Q_0 not containing b_{01}. The Q_1, Q_2 are lifted using β, which subdivides the primary cell (Figure 8). This subdivision "coincides" with the restriction in $c_{01}+Q_1+Q_2$ of the subdivision by β, except that the latter uses c_{011} whereas the former uses b_{01}, i.e. the integer points in both subdivisions are the same and are assigned the same RC.

• We study the Recursion Phase on secondary cell:

$$\mathcal{F}_{v_1} = \mathrm{CH}(b_{01}, k_0 F_{v_1}) + k_1 F_{v_1} + k_2 F_{v_1},$$

defined by facet $F_{v_1} = ((0,1),(0,2)) \subset Q$ supported by v_1, see Figure 10. Now,

$$A_{1v_1} = \{(0,2),(0,4)\},\ \ A_{2v_1} = \{(0,1),(0,2)\},$$

and the lattice generated by $A_{1v_1}+A_{2v_1}$ is $L_+ := \langle(0,3),(0,4)\rangle \cong L_{v_1} \cong \mathbb{Z}$. The index of L_+ in L_{v_1} is $\mathrm{ind}_{v_1}=1$ and the coset representative for L_+ in L_{v_1} is $q_0=(0,0)$. The v_1-*lattice diameter* is

$$d_{v_1} := b_{01}\cdot v_1 - \min_{p\in \mathrm{CH}(b_{01},k_0F_{v_1})} p\cdot v_1 = 1.$$

Hence, there is one slice corresponding to one piece. We describe the recursion step on this piece. It

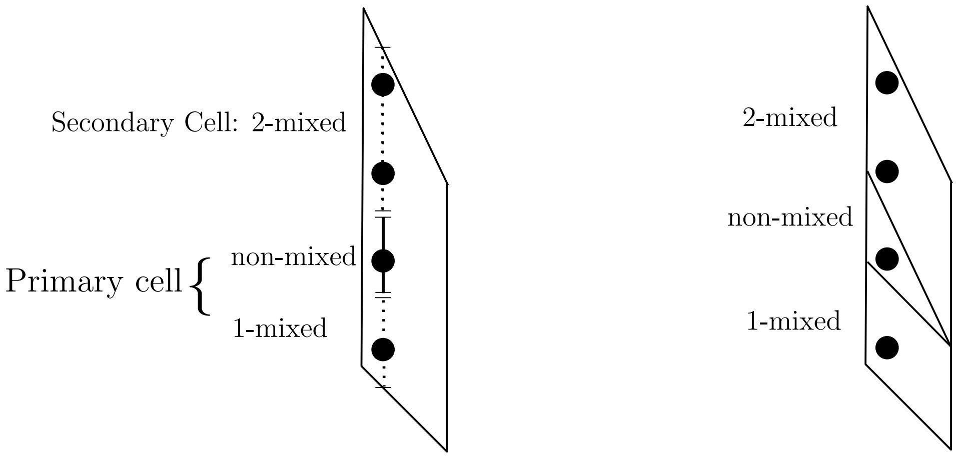

Figure 10: Example 22: The piece of the secondary cell $\mathcal{F}_{v_1}$ w.r.t. vector $v_1=(1,0)$ and its mixed subdivision (left). Also drawn is the corresponding secondary cell and its mixed subdivision w.r.t Alg. B (right)

contains points corresponding to $(0,4),(0,5),\ (0,6),\ (0,7)$ lying on the slice of $\mathcal{F}_{v_1}+\delta$ of the form

$$(\tilde{\lambda}k_0F_{v_1}+\delta') + k_1F_{v_1} + k_2F_{v_1} + \lambda F_{v_1} + \delta.$$

To define the piece, following notation in [D'A02], the scalar multiple of F_{v_1} is $\tilde{\lambda}F_{v_1} = \frac{29}{30}F_{v_1}$ and the translation vector is $\delta' := (\frac{1}{30},0)$. Since we do not use an initial additional polytope, $\lambda=0$ and $\lambda_{v_1} := \lambda+\tilde{\lambda} = \frac{29}{30}$.

Let $\delta_\lambda := \delta+\delta' = (0,-\frac{1}{30})$, and $\delta_\lambda = \delta_\lambda^{v_1} + \delta_{\lambda v_1}$, where $\delta_\lambda^{v_1} = (0,0) \in \mathbb{Q}v_1$ and $\delta_{\lambda v_1} = (0,-\frac{1}{30}) \in L_+\otimes\mathbb{Q}$, hence $\delta_{0v_1} := \delta_{\lambda v_1} - q_0 = (0,-\frac{1}{30})$. So, the slice of $\mathcal{F}_{v_1}+\delta$ is

$$k_1F_{v_1} + k_2F_{v_1} + \lambda_{v_1}k_0F_{v_1} + \delta_\lambda, \tag{23}$$

and the corresponding piece in L_+ is

$$k_1F_{v_1} + k_2F_{v_1} + \lambda_{v_1}k_0F_{v_1} + \delta_{0v_1}. \tag{24}$$

The bijection between points in (23) and (24) is

$$p = \bar{p} + \delta_\lambda^{v_1} + q_0 = \bar{p},$$

where $p \in$ (23) and $\bar{p} \in$ (24). After re-indexing, the input of the recursion step is:
- the polygons $\overline{Q_0} := k_1 F_{v_1}$, $\overline{Q_1} := k_2 F_{v_1}$, and $\overline{Q_2} := \frac{29}{30} k_0 F_{v_1}$ which is the additional polytope,
- the lattice $L^{(1)} := L_+ = \langle (0,3), (0,4) \rangle$ and
- the perturbation vector $\overline{\delta_0} := \delta_{0v_1} = (0, -\frac{1}{30})$.

In order to be compatible with β, we choose $\overline{b_{01}} = b_{12} = (0,2)$ and apply the primary lifting. This partitions $\overline{Q_0} + \overline{Q_1} + \overline{Q_2} + \overline{\delta_0}$ into a primary $\overline{b_{01}} + \overline{Q_1} + \overline{Q_2} + \overline{\delta_0}$ and a secondary cell $\overline{Q_0} + (0,2) + \frac{29}{30}(0,2) + \overline{\delta_0}$. Lifting β induces a mixed subdivision on the primary cell consisting of the cells $\overline{b_{01}} + (0,1) + \overline{Q_2} + \overline{\delta_0}$ and $\overline{b_{01}} + \overline{Q_1} + \frac{29}{30}(0,1) + \overline{\delta_0}$. The former is non-mixed and contains point $(0,5)$, corresponding to the same point on the slice, which is also non-mixed under Alg. B. The latter cell is $\overline{0}$-mixed, hence 1-mixed and contains point $(0,4)$, corresponding to the same point on the slice, which is also 1-mixed under Alg. B. The secondary cell $\overline{Q_0} + (0,2) + \frac{29}{30}(0,2) + \overline{\delta_0}$ is $\overline{1}$-mixed, hence 2-mixed and contains the integer points $(0,6), (0,7)$ corresponding to the same points on the slice. They are also 2-mixed under Alg. B.

• We apply recursion on secondary cell:

$$\mathcal{F}_{v_2} = \mathrm{CH}(b_{01}, k_0 F_{v_2}) + k_1 F_{v_2} + k_2 F_{v_2},$$

defined by the facet $F_{v_2} = ((0,2),(1,2))$ of Q supported by v_2, see Figure 11. Now,

$$A_{1v_2} = \{(0,4),(2,4)\}, \; A_{2v_2} = \{(0,2),(1,2)\}$$

and the lattice generated by $A_{1v_2} + A_{2v_2}$ is $L_+ := \langle (0,6),(1,6) \rangle \cong L_{v_2} \cong \mathbb{Z}$. The index of L_+ in L_{v_2} is $\mathrm{ind}_{v_2} = 1$ and the coset representative for L_+ in L_{v_2} is $q_0 = (0,0)$. The v_2-*lattice diameter* is

$$d_{v_2} := b_{01} \cdot v_2 - \min_{p \in \mathrm{CH}(b_{01}, k_0 F_{v_2})} p \cdot v_2 = 2.$$

Hence, there are two slices, each containing one piece, and the algorithm recurses on each such piece.

We analyze the recursion step on the piece of the shifted secondary cell $\mathcal{F}_{v_2} + \delta$, which contains the integer points corresponding to the points $(1,7), (2,7), (3,7)$ lying on a slice of the shifted secondary cell $\mathcal{F}_{v_2} + \delta$ of the form

$$(\tilde{\lambda} k_0 F_{v_2} + \delta') + k_1 F_{v_2} + k_2 F_{v_2} + \lambda F_{v_2} + \delta.$$

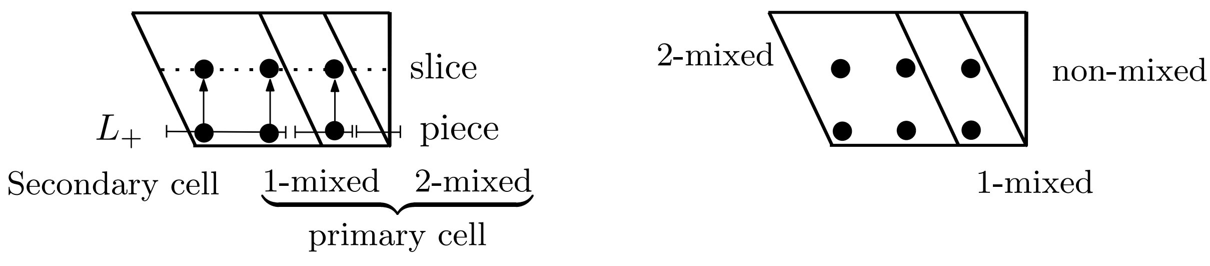

Figure 11: Example 22: A slice of the secondary cell $\mathcal{F}_{v_2}$ w.r.t. vector $v_2 = (0,-1)$ containing points $(1,7), (2,7), (3,7)$ (dotted segment, left subfigure), the corresponding piece and its mixed subdivision w.r.t. Alg. A. The arrows show the correspondence between points on the slice and points on the piece. Also depicted is the mixed subdivision of the corresponding secondary cell w.r.t. Alg. B (right subfigure)

To define this piece we have that F_{v_2} is $\tilde{\lambda} F_{v_2} = \frac{31}{60} F_{v_2}$ and the translation vector $\delta' := (\frac{29}{60}, 0)$. Now $\lambda = 0$ and hence $\lambda_{v_2} := \lambda + \tilde{\lambda} = \frac{31}{60}$. Let $\delta_\lambda := \delta + \delta' = (\frac{9}{29}, -\frac{1}{30})$. Then, δ_λ can be written as $\delta_\lambda = \delta_\lambda^{v_2} + \delta_{\lambda v_2}$, where $\delta_\lambda^{v_2} = (0,1) \in \mathbb{Q}v_2$ and $\delta_{\lambda v_2} = (\frac{9}{20}, -\frac{31}{30}) \in L_+ \otimes \mathbb{Q}$, hence $\delta_{0v_2} := \delta_{\lambda v_2} - q_0 = (\frac{9}{20}, -\frac{31}{30})$.

So, the slice of $\mathcal{F}_{v_2} + \delta$ is

$$k_1F_{v_2} + k_2F_{v_2} + \lambda_{v_2}k_0F_{v_2} + \delta_\lambda, \tag{25}$$

and the corresponding piece in L_+ is

$$k_1F_{v_2} + k_2F_{v_2} + \lambda_{v_2}k_0F_{v_2} + \delta_{0v_2}. \tag{26}$$

The bijection between points in (25) and points in (26) is

$$p = \bar{p} + \delta_\lambda^{v_2} + q = \bar{p} + (0,1),$$

where $p \in (25)$ and $\bar{p} \in (26)$.

After re-indexing, the input of the recursion step is:
- the polygons $\overline{Q_0} := k_1F_{v_2}$, $\overline{Q_1} := k_2F_{v_2}$, and $\overline{Q_2} := \frac{31}{60}k_0F_{v_2}$ which is the additional polytope,
- the lattice $L^{(1)} := L_+ = \langle(0,6),(1,6)\rangle$ and
- the perturbation vector $\bar{\delta} := \delta_{0v_2} = (\frac{9}{20}, -\frac{31}{30})$.

To be compatible with β, we choose $\overline{b_{01}} = b_{14} = (2,4)$ and apply the primary lifting; this partitions the Minkowski sum $\overline{Q_0} + \overline{Q_1} + \overline{Q_2} + \bar{\delta}$ into a primary $\overline{b_{01}} + \overline{Q_1} + \overline{Q_2} + \bar{\delta}$ and a secondary cell $\overline{Q_0} + (0,2) + \frac{31}{60}(0,2) + \bar{\delta}$. Lifting β induces a mixed subdivision of the primary cell consisting of the cells $\overline{b_{01}} + (1,2) + \overline{Q_2} + \bar{\delta}$ and $\overline{b_{01}} + \overline{Q_1} + \frac{31}{60}(0,2) + \bar{\delta}$. The latter is $\overline{0}$-mixed, hence 1-mixed and contains the integer point $(3,6)$ corresponding to point $(3,7)$ on the slice which is also 1-mixed under Alg. B. The former is non-mixed and does not contain any integer points.

The secondary cell $\overline{Q_0} + (0,2) + \frac{31}{60}(0,2) + \bar{\delta}$ is $\overline{1}$-mixed, hence 2-mixed and contains the integer points $(1,6),(2,6)$ corresponding to the points $(1,7),(2,7)$ of the slice respectively; they are also 2-mixed under Alg. B.

- The last secondary cell is

$$\mathcal{F}_{v_3} = \mathrm{CH}(b_{01}, k_0F_{v_3}) + k_1F_{v_3} + k_2F_{v_3},$$

defined by the facet $F_{v_3} = ((3,0),(1,2))$ of Q supported by $v_3 = (-1,-1)$., see also Figure 6 and Example 16. Now,

$$A_{1v_3} = \{(6,0),(2,4)\}, A_{2v_3} = \{(3,0),(1,2)\},$$

the lattice generated by $A_{1v_3} + A_{2v_3}$ is $L_+ := \langle(9,0),(7,2)\rangle \cong 2\mathbb{Z}$ and $L_{v_3} \cong \mathbb{Z}$. The index of L_+ in L_{v_3} is $\mathrm{ind}_{v_3} = 2$ and the cosets representatives for L_+ in L_{v_3} are $q_0 = (0,0)$ and $q_1 = (-1,1)$. The v_3-*lattice diameter* is

$$d_{v_3} := b_{01} \cdot v_3 - \min_{p \in \mathrm{CH}(b_{01}, k_0F_{v_3})} p \cdot v_3 = 2.$$

Hence there are two slices, each corresponding to two pieces, and the algorithm recurses on each such piece.

We analyze the recursion step on the two pieces that contain integer points corresponding to points $(11,0),(10,1),(9,2)$, $(8,3),(7,4)$, $(6,5),(5,6)$, $(4,7)$ lying on a slice of the shifted secondary cell $\mathcal{F}_{v_3} + \delta$ of the form

$$(\tilde{\lambda}k_0F_{v_3} + \delta') + k_1F_{v_3} + k_2F_{v_3} + \lambda F_{v_3} + \delta.$$

To define these pieces, we have that the scalar multiple of F_{v_3} is $\tilde{\lambda}F_{v_3} = \frac{32}{60}F_{v_3}$ and the translation vector is $\delta' := (\frac{7}{15}, 0)$. Now, $\lambda = 0$ and hence $\lambda_{v_3} := \lambda + \tilde{\lambda} = \frac{32}{60}$; Let $\delta_\lambda := \delta + \delta' = (\frac{13}{30}, -\frac{1}{30})$.

Then, δ_λ can be written as $\delta_\lambda = \delta_\lambda^{v_3} + \delta_{\lambda v_3}$, where $\delta_\lambda^{v_3} = (1,1) \in \mathbb{Q}v_3$ and $\delta_{\lambda v_3} = (-\frac{17}{30}, -\frac{31}{30}) \in L_+ \otimes \mathbb{Q}$, hence $\delta_{0v_3} := \delta_{\lambda v_3} - q_0 = (-\frac{17}{30}, -\frac{31}{30})$ and $\delta_{1v_3} := \delta_{\lambda v_3} - q_1 = (\frac{13}{30}, -\frac{61}{30})$.

So, the slice of $\mathcal{F}_{v_3} + \delta$ is

$$k_1F_{v_3} + k_2F_{v_3} + \lambda_{v_3}k_0F_{v_3} + \delta_\lambda, \tag{27}$$

Table 1: Illustration of Cor. 19 and Cor. 20 for Example 22

Cell w.r.t. Alg. A	Corresponding cell w.r.t. Alg. B	Type of cell
$\tilde{\lambda}(1,2)+(6,0)+((3,0),(1,2))+\delta_{0v3}$	$(c_{011},(1,2))+c_{154}+((3,0),(1,2))+\delta$	1-mixed
$\tilde{\lambda}((3,0),(1,2))+(6,0)+(3,0)+\delta_{0v3}$	$\mathrm{CH}(c_{011},(1,2),(3,0))+c_{154}+(3,0)+\delta$	non-mixed
$\tilde{\lambda}(1,2)+(6,0)+((3,0),(1,2))+\delta_{1v3}$	$(c_{011},(1,2))+c_{154}+((3,0),(1,2))+\delta$	1-mixed
$\tilde{\lambda}((3,0),(1,2))+(6,0)+(3,0)+\delta_{1v3}$	$\mathrm{CH}(c_{011},(1,2),(3,0))+c_{154}+(3,0)+\delta$	non-mixed
$\tilde{\lambda}(0,2)+(2,4)+((0,2),(1,2))+\delta_{0v2}$	$(c_{011},(0,2))+c_{143}+((0,2),(1,2))+\delta$	1-mixed
$\tilde{\lambda}((0,2),(1,2))+(2,4)+(1,2)+\delta_{0v2}$	$\mathrm{CH}(c_{011},(1,2),(0,2))+c_{143}+(1,2)+\delta$	non-mixed

and the corresponding pieces in L_+ are

$$k_1F_{v_3}+k_2F_{v_3}+\lambda_{v_3}k_0F_{v_3}+\delta_{0v_3}, \tag{28}$$

$$k_1F_{v_3}+k_2F_{v_3}+\lambda_{v_3}k_0F_{v_3}+\delta_{1v_3}, \tag{29}$$

The correspondences between points in the slice and points in the pieces are

$$p=\bar{p}+\delta_\lambda^{v_3}+q_0=\bar{p}+(1,1),$$

where $p\in(27)$ and $\bar{p}\in(28)$, and

$$p=\bar{p}+\delta_\lambda^{v_3}+q_1=\bar{p}+(0,2),$$

where $p\in(27)$ and $\bar{p}\in(29)$.

After re-indexing, the input of the recursion step is:
- the polygons $\overline{Q_0}:=k_1F_{v_3}$, $\overline{Q_1}:=k_2F_{v_3}$, and $\overline{Q_2}:=\frac{32}{60}k_0F_{v_3}$ which is the additional polytope,
- the lattice $L^{(1)}:=L_+=\langle(9,0),(7,2)\rangle$ and
- the perturbation vectors $\overline{\delta_0}:=\delta_{0v_3}=(-\frac{17}{30},-\frac{31}{30})$ and $\overline{\delta_1}:=\delta_{1v_3}=(\frac{13}{60},-\frac{61}{30})$.

As β indicates, we choose $\overline{b_{01}}=b_{15}=(6,0)$ and apply the primary lifting.

For the first piece, the lifting partitions the Minkowski sum $\overline{Q_0}+\overline{Q_1}+\overline{Q_2}+\overline{\delta_0}$ into a primary $\overline{b_{01}}+\overline{Q_1}+\overline{Q_2}+\overline{\delta_0}$ and a secondary cell $\overline{Q_0}+(1,2)+\frac{32}{60}(1,2)+\overline{\delta_0}$. Lifting β induces a mixed subdivision on the primary cell consisting of the cells $\overline{b_{01}}+(3,0)+\overline{Q_2}+\overline{\delta_0}$ and $\overline{b_{01}}+\overline{Q_1}+\frac{32}{60}(1,2)+\overline{\delta_0}$. The former is non-mixed and contains point $(9,0)$, which corresponds to $(10,1)$ on the slice which is also non-mixed under Alg. B. The latter is $\overline{0}$-mixed, hence 1-mixed and contains the point $(7,2)$ corresponding to the point $(8,3)$ in the slice which is also 1-mixed under Alg. B.

The secondary cell $\overline{Q_0}+(1,2)+\frac{32}{60}(1,2)+\overline{\delta_0}$ is $\overline{1}$-mixed, hence 2-mixed and contains the integer points $(3,6),(5,4)$ corresponding to the points $(4,7),(6,5)$ of the slice respectively which are also 2-mixed under Alg. B.

For the second piece, the lifting partitions the Minkowski sum $\overline{Q_0}+\overline{Q_1}+\overline{Q_2}+\overline{\delta_1}$ into a primary $\overline{b_{01}}+\overline{Q_1}+\overline{Q_2}+\overline{\delta_1}$ and a secondary cell $\overline{Q_0}+(1,2)+\frac{32}{60}(1,2)+\overline{\delta_1}$. Lifting β induces a mixed subdivision on the primary cell consisting of the cells $\overline{b_{01}}+(3,0)+\overline{Q_2}+\overline{\delta_1}$ and $\overline{b_{01}}+\overline{Q_1}+\frac{32}{60}(1,2)+\overline{\delta_1}$. The former is non-mixed and contains point $(11,-2)$ corresponding to $(11,0)$ on the slice which is also non-mixed under Alg. B, whereas the latter cell is $\overline{0}$-mixed, hence 1-mixed and contains the integer point $(9,0)$ corresponding to point $(9,2)$ on the slice which is also 1-mixed under Alg. B.

The secondary cell $\overline{Q_0}+(1,2)+\frac{32}{60}(1,2)+\overline{\delta_1}$ is $\overline{1}$-mixed, hence 2-mixed and contains the integer points $(7,2),(5,4)$ corresponding to the points $(7,4),(5,6)$ of the slice respectively. These are also 2-mixed under Alg. B.

The second slice of $\mathcal{F}_{v_3}+\delta$ is $\left(\frac{1}{30}F_{v_3}+(\frac{29}{30},0)\right)+k_1F_{v_3}+k_2F_{v_3}+(-\frac{1}{30},-\frac{1}{30})$, and contains integer points $(10,0),(9,1),(8,2),(7,3),(6,4),(5,5),(4,6)$.

Table 1 illustrates corollaries 19 and 20, where the summands come from Q_0,Q_1 and Q_2 respectively. Recall that $c_{011}:=(1,0)+\delta_{011}$, $c_{143}:=(2,4)+\delta_{143}$ and $c_{154}:=(6,0)+\delta_{154}$.

6 Further work

Let us conclude with some preliminary results on mixed algebraic systems. In studying systems with different Newton polytopes, we need the following:

Definition 23. The set of polytopes $Q_1, \ldots, Q_h \subset \mathbb{R}^n$, s.t. $\dim(\langle Q_1, \ldots, Q_h \rangle) = h - 1$, is *essential* if every subset of cardinality $j, 1 \leq j < h$ spans a space of dimension $\geq j$.

The toric resultant is well defined only for essential sets of Newton polytopes. An essential set defines a Minkowski sum of dimension $h-1$ but the converse is not always true.

Alg. A admits one main modification in the mixed case: At the Recursion Phase, the faces $F_i \subset Q_i$ supported by vector v are not always the same. Let the input be $n+1$ polytopes; we describe the 0-th iteration for simplicity. Consider the n-dimensional secondary cell:

$$\mathrm{CH}(b_{01}, F_0) + F_1 + \cdots + F_n \subset \mathbb{R}^n,$$

where $F_i \subset \mathbb{R}^{n-1}$. Without loss of generality, let $\{F_1, \ldots, F_k\}$ be an essential subset and let $L_+(k)$ be the integer lattice it defines. The algorithm recurses on lattice $L_+(k)$ and polytope set (representing a piece)

$$\mathrm{CH}(b_{01}, F_0) \cap \Lambda_+(k), F_1, \ldots, F_k, F_{k+1} \cap \Lambda_+(k), \ldots, F_n \cap \Lambda_+(k), \tag{30}$$

where $\Lambda_+(k)$ ranges over all possible homothetic copies of $L_+(k)$ defined by the different cosets of $L_+(k)$ in its saturation, and the different slices that can be defined as intersections with $\mathrm{CH}(b_{01}, F_0)$. Alg. A distinguishes two cases, according to whether there is one or more essential subsets of $\{F_1, \ldots, F_n\}$. In the former case, v and the corresponding secondary cell are called *admissible*. For non-admissible cells, all integer points are considered as non-mixed, i.e. treated as if they lied in non-mixed cells. For admissible cells, integer d_{F_v} is defined [D'A02, Sec.4] (cf. [Min03]), and d_{F_v} pieces of the form (30) are (arbitrarily) selected. Lattice points labeled as mixed in these pieces by the recursive application of Alg. A are labeled as mixed overall, the rest are non-mixed.

Before sketching the extension of our algorithm to the mixed case, let us consider some special cases. Reduced systems are such that, for any vector $v \in \mathbb{R}^n$, there is some $i \in \{1, \ldots, n\}$ so that the face supported by v in Q_i is a vertex [D'A01]. For us, it suffices that this holds for any vector v associated with secondary cells at the 0-th recursion step of Alg. A. For such systems, as well as for arbitrary systems of three bivariate polynomials ($n = 2$), the lifting function (31) produces a Macaulay-type formula [DE03a].

$$\begin{array}{llll} l_0: & A_0 & \to & \{0,1\} \\ & b_{01} & \mapsto & 1, \\ & b_{0j} & \mapsto & 0, \quad \text{if } j \neq 0, \end{array} \qquad \begin{array}{llll} l_i: & A_i & \to & \mathbb{R} \quad (i \geq 1) \\ & p & \mapsto & 0, \quad \text{if } p \notin \cup_{\forall v} A_{i,v} \\ & p & \mapsto & r_p \quad \text{otherwise.} \end{array} \tag{31}$$

Here, $A_{i,v} := A_i \cap Q_{i,v}$, where $Q_{i,v}$ is the face of Q_i supported by v, and r_p is a positive random number satisfying $0 < r_p \ll 1$. It is not difficult to see that our lifting β has an overall effect similar to that of lifting (31), therefore it also produces a Macaulay-type formula for the previous systems. For bivariate systems, the idea of the proof is subsumed by that for $n = 3$ at the end of this section.

For extending Alg. B to the mixed case, we must modify it so that Definition 5 applies to different polytopes and also up to $i = n-1$. We sketch a proof that it produces the same matrix as Alg. A, by extending the correlation between maximal cells, established in the unmixed case. Our proof might extend to $n > 3$, but seems complicated; we hope that a more elegant approach is possible.

In non-admissible secondary cells of Alg. A, for any n, we show that both algorithms behave in the same way, namely that the corresponding lattice points lie in non-mixed cells of Alg. B. We demonstrate the contrapositive by focusing on a mixed cell of Alg. B and a corresponding secondary cell of Alg. A, following Lemma 18.

Lemma 24. *Every t-mixed cell by Alg. B, when intersected with a $(n-t)$-dimensional hyperplane as in Lemma 18, is contained in an admissible secondary cell of step $t-1$ of Alg. A.*

Proof. Any t-mixed cell of Alg. B is of the form $E_0 + \cdots + E_{t-1} + a_{tj} + E_{t+1} + \cdots + E_n$, where a_{tj} is either a vertex of Q_i or some c_{tjs} in the interior of an $(n-t)$-dimensional face, and edges $E_{t+1}, \ldots, E_n$ span an $(n-t)$-dimensional space. This cell is intersected by a $(n-t)$-dimensional hyperplane, similarly to Lemma 18. The intersection is contained in a t-primary cell of Alg. A with t-summand b_{tj}; it lies in a piece of $(t-1)$-secondary cell

$$F_0 + \cdots + F_{t-2} + \mathrm{CH}(b_{(t-1)h}, F_{t-1}) + F_t + \cdots + F_n,$$

where the F_i are faces of the Q_i, $i = 1, \ldots, n$, supported by the same vector, with $\dim F_i \leq n-t$. We claim $\{F_t, \ldots, F_n\}$ contains a unique essential set, with cardinality $r+1$, spanning an r-dimensional space, which is defined as follows: F_t and $r \leq n-t$ faces, denoted, without loss of generality, $F_{t+1}, \ldots, F_{t+r}$, where r is minimal so that $\dim H = r$, for $H = \langle F_t, \ldots, F_{t+r} \rangle$.

By hypothesis, $\dim\langle F_{t+1}, \ldots, F_n \rangle = n-t$, since a subspace is spanned by the E_i and has same dimension. So subsets indexed in $\{t+1, \ldots, n\}$ span a space of dimension at least equal to their cardinality. In addition, none of the $F_i, i > t+r$ is contained in H. So every subset indexed in $\{t, \ldots, n\}$ containing $\{t\} \cup J$, for $J \subset \{t+r+1, \ldots, n\}$, will be of cardinality $\leq r + |J|$ and span a space of dimension $r + |J|$. Hence there are no other essential subsets. □

For $n = 3$, all admissible secondary cells have d_{F_v} pieces, since there is no extra artificial polytope in the input of Alg. A. We distinguish cases on the dimension $k-1$ of the space generated by the essential set $\{F_1, \ldots, F_k\}, 1 \leq k \leq 3$, on which the recursion of Alg. A occurs:

1. If $k-1$ is 0 or 1, the recursion is either trivial (occurs on a vertex), or corresponds to the Sylvester case.

2. If $k-1 = 2$ and $\dim F_i = 1, i = 1, 2, 3$, the two algorithms behave similarly, since Definition 5 defines points c_{2js} in the edges of Q_2 and Lemma 18 applies. Notice that $\dim Q_2 \geq 1$; otherwise the Q_i's would not form an essential set.

3. If $k-1 = 2$, then $\dim F_i \in \{1, 2\}$ for $i = 1, 2, 3$ and at least one face is two-dimensional. If $\dim F_1 = 2$, then Lemma 18 applies. Otherwise, $\dim F_1 = 1$ and $\dim F_2 \geq 1$. Irrespective of $\dim F_2$, the c_{2js}'s play the role of distinguished points and Lemma 18 applies again.

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