# Computing the Newton Polygon of the Implicit Equation

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**Abstract.** We consider rationally parameterized plane curves, where the polynomials in the parameterization have fixed supports and generic coefficients. We apply sparse (or toric) elimination theory in order to determine the vertex representation of the implicit equation's Newton polygon. In particular, we consider mixed subdivisions of the input Newton polygons and regular triangulations of point sets defined by Cayley's trick. We consider polynomial and rational parameterizations, where the latter may have the same or different denominators; the implicit polygon is shown to have, respectively, up to 4, 5, or 6 vertices.

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**Keywords.** Sparse (toric) resultant, implicitization, Newton polygon, Cayley trick, mixed subdivision.

## 1. Introduction

Implicitization is the problem of switching from a parametric representation of a hypersurface to an algebraic one. It is a fundamental question with several applications. Here we consider the implicitization problem for a planar curve, where the polynomials in its parameterization have fixed Newton polytopes. We determine the vertices of the Newton polygon of the implicit equation, or *implicit polygon*, without computing the equation, under the assumption of *generic* coefficients relative to the given supports, i.e. our results hold for all coefficient vectors in some open dense subset of the coefficient space. The support of the implicit equation, or *implicit support*, is taken to be all interior points inside the implicit polygon.

This problem was posed in [SY94] but has received much attention lately. According to [STY07], "a priori knowledge of the Newton polytope would greatly facilitate the subsequent computation of recovering the coefficients of the implicit equation [...] This is a problem of numerical linear algebra ...". Reducing implicitization to linear algebra is also the premise of [CGKW01, EK03]. Of course, this can be nontrivial if coefficients are not generic. Another potential application of knowing the implicit polygon is to approximate implicitization, see [Dok01].

Our approach considers the symbolic resultant which eliminates the parameters and, then, is specialized to yield an equation in the implicit variables. This method applies, more generally, to applications, including the computation of the $u$-resultant or the offset of a parametric curve or surface, where the resultant coefficients are polynomials in a few variables, and we wish to study the resultant as a polynomial in these variables.

Previous work includes [EK03, EK05], where an algorithm constructs the Newton polytope of any implicit equation. That method computes all mixed subdivisions, then applies cor. 3. In [GKZ94, chapter 12], they study the resultant of two univariate polynomials and describe the facets of its Newton polytope. In [GKZ90], the extreme monomials of the Sylvester resultant are described. The approaches in [EK03, GKZ94] cannot exploit the fact that the denominators in a rational parameterization may be identical.

Tropical geometry can also give the implicit polytope of any hypersurface parameterized by Laurent polynomials [STY07, DFS07]. In [SY07], they consider this polytope as the *mixed fiber polytope* of the input polytopes, while software TrIm computes it. For curves, the implicit polygon is described in [STY07, example 1.1], without any hypothesis of genericity. The approach handles rational parameterizations with the same denominator by homogenizing the parameter as well as the implicit space. This theory extends to arbitrary implicit ideals. In [EK07] the problem was solved in the context of elimination theory by means of composite bodies and mixed fiber polytopes.

Another independent line of work applies a refinement of the Bernstein-Kushnirenko-Khovanskii (BKK) bound on the number of isolated roots of a polynomial system in the torus [PS07]. Instead of dealing explicitly with the implicit polygon, the authors of [DS07] study its support function, which completely characterizes it, and reduce the problem to counting the number of solutions of a certain system of equations. Eventually, they obtain the normal fan of the implicit polygon from the root multiplicities of the polynomials in any parameterization of the rational plane curve. Their main theorem does not rely on any genericity condition and implies an algorithm to compute the implicit polygon of any parametric curve. They also address the question of deciding for a given polygon whether it can be the Newton polygon of an implicit curve. They show that every non-degenerate polygon is the Newton polygon of a rational plane curve, that the variety of rational curves with given Newton polygon is unirational, and they compute its degree.

In [EKP07], we computed the Newton polytope of specialized resultants while avoiding to compute the entire secondary polytope; our approach was to examine the silhouette of the latter with respect to an orthogonal projection. This method is revisited in [EFK10] by studying output-sensitive methods to compute the resultant polytope. A survey of the recent results in the area can be found in [DS09].

The main contribution of this paper is to determine the vertex structure of the implicit polygon of a rational parameterized planar curve, or implicit vertices, under the assumption of *generic* coefficients. If the coefficients are not sufficiently generic, then the computed polygon contains the implicit polygon. In the case of rationally parameterized curves with different denominators (which includes the case of Laurent polynomial parameterizations), the Cayley trick reduces the problem to computing regular triangulations of point sets in the plane. If the denominators are identical, two-dimensional mixed subdivisions are examined; we show that only subdivisions obtained by *linear* liftings are relevant. These results also apply if the two parametric expressions share the same numerator, or the numerator of one equals the denominator of the other. We prove that, in all these cases, only extremal terms matter in determining the implicit polygon as well as in ensuring the genericity hypothesis on the coefficients. Our presentation is self-contained; our methods are independent as well as different from other approaches. In [EKP07] some of these results, mainly for the case of different denominators, were presented in preliminary form.

The following proposition collects our main corollaries regarding the shape of the implicit polygon in terms of corner cuts on an initial polygon. A corner cut on a polygon $P$ is a line that intersects the polygon, excluding one vertex while leaving the rest intact. $\phi$ is the implicit equation and $\mathcal{N}(\phi)$ is the implicit polygon.

**Proposition 1.** *$\mathcal{N}(\phi)$ is a polygon with one vertex at the origin and two edges lying on the axes. In particular, for polynomial parameterizations, $\mathcal{N}(\phi)$ is a right triangle with at most one corner cut, which excludes the origin. For rational parameterizations with equal denominators, $\mathcal{N}(\phi)$ is a right triangle with at most two cuts, on the same or different corners. For rational parameterizations with different denominators, $\mathcal{N}(\phi)$ is a quadrilateral with at most two cuts, on the same or different corners.*

**Example 1.** Consider the plane curve parameterized by:

$$x = \frac{t^6+2t^2}{t^7+1}, y = \frac{t^4-t^3}{t^7+1},$$

Theorem 20 yields vertices $(7,0),(0,7),(0,3),(3,1),(6,0)$, which define the actual implicit polygon (see figure 1, left) because the implicit equation is

$$\begin{aligned}\phi = &-32y^4 - 30x^3y^2 - x^4y - 12x^2y^2 - 3x^3y - 7x^6y - 2x^7 + 20xy^3 + 280x^2y^5 \\ &-7^3y^4x - 70x^4y^3 - 22x^3y^3 - 49x^5y^2 - 21x^4y^2 + 11x^5y + 216y^5 + 129y^7 \\ &-248y^6 + 70xy^6 + 185xy^5 + 24y^3 + 100xy^4 + 43x^2y^3 + 72x^2y^4 + 3x^6.\end{aligned} \tag{1}$$

Changing the coefficient of $t^2$ to -1, leads to an implicit polygon with 4 cuts which is contained in the polygon predicted by theorem 20. This shows the importance of the genericity condition on the coefficients of the parametric polynomials. See example 11 for details.

An instance where the implicit polygon has 6 vertices is:

$$x = \frac{t^3 + 2t^2 + t}{t^2 + 3t - 2}, \ y = \frac{t^3 - t^2}{t - 2}.$$

Our results in section 3 yield implicit vertices $(0, 1), (0, 3), (3, 0), (1, 3), (2, 0), (3, 2)$ which define the actual implicit polygon (see figure 1, right). See example 5 for details.

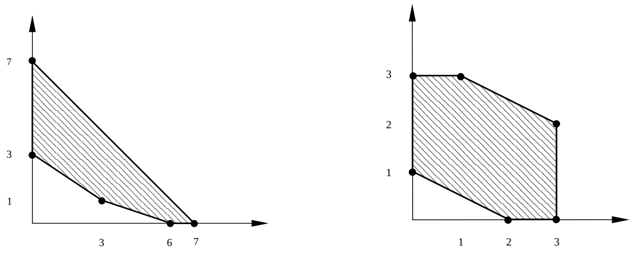


FIGURE 1. The implicit polygons of the curves of example 1.

The paper is organized as follows. The next section recalls concepts from sparse elimination and focuses on the Newton polytope of the sparse resultant. It also defines the problem of computing the implicit polytope. Section 3 refers to rational parametric curves, where denominators are different. The problem is reduced to studying triangulations of point sets in the plane. Section 4 solves the problem for rational parameterizations with identical denominators, by studying relevant mixed subdivisions. We conclude with further work in section 5.

## 2. Sparse elimination and Implicitization

We first recall some notions of sparse elimination theory; see [GKZ94] for more information. Then, we define the problem of implicitization.

Given a polynomial $f$, its *support* $\mathcal{A}(f)$ is the set of the exponent vectors corresponding to monomials with nonzero coefficients. Its *Newton polytope* $\mathcal{N}(f)$ is the convex hull of $\mathcal{A}(f)$, denoted $\mathrm{CH}(\mathcal{A}(f))$. The *Minkowski sum* $A + B$ of (convex polytopes) $A, B \subset \mathbb{R}^n$ is the set $A + B = \{a + b \mid a \in A, b \in B\} \subset \mathbb{R}^n$.

**Definition 1.** Consider Laurent polynomials $f_i, \ i = 0, \ldots, n$, in $n$ variables, with fixed supports. Let $\mathrm{c} = (c_{0,0}, \ldots, c_{0,s_0}, \ldots, c_{n,0}, \ldots, c_{n,s_n})$ be the vector of all nonzero (symbolic) coefficients. The sparse (or toric) resultant $\mathcal{R}$ of the $f_i$ is the unique, up to sign, irreducible polynomial in $\mathbb{Z}[\mathrm{c}]$, which vanishes iff the $f_i$ have a common root in the toric variety corresponding to the supports of the $f_i$.

Let the system's Newton polytopes be $P_0, \ldots, P_n \subset \mathbb{R}^n$. Their *mixed volume* is the unique integer-valued function, which is symmetric, multilinear with respect to Minkowski addition, and satisfies $\mathrm{MV}(Q, \ldots, Q) = n! \ \mathrm{Vol}(Q)$, for any lattice polytope $Q \subset \mathbb{R}^n$, where $\mathrm{Vol}(\cdot)$ indicates Euclidean volume. We shall abuse notation and denote the mixed volume of a family of supports $A_0, \ldots, A_n$ by $\mathrm{MV}(A_0, \ldots, A_n)$ instead of $\mathrm{MV}(\mathrm{CH}(A_0), \ldots, \mathrm{CH}(A_n))$. For the rest of the paper we assume that the Minkowski sum $P = P_0 + \cdots + P_n \subset \mathbb{R}^n$ is $n$-dimensional. The family of supports $A_0, \ldots, A_n$ is *essential* according to the terminology of [Stu94, sec. 1]. This is equivalent to the existence of a non-zero partial mixed volume $\mathrm{MV}_i = \mathrm{MV}(A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$, for some $i \in \{0, \ldots, n\}$

A *Minkowski cell* of $P$ is any full-dimensional convex polytope $B = \sum_{i=0}^n B_i$, where each $B_i$ is a convex polytope with vertices in $A_i$. We say that two Minkowski cells $B = \sum_{i=0}^n B_i$ and $B' = \sum_{i=0}^n B'_i$ intersect properly when the intersection of the polytopes $B_i$ and $B'_i$ is a face of both and their Minkowski sum descriptions are compatible, cf [San05].

**Definition 2.** [San05, definition 1.1] A *mixed subdivision* of $P$ is any family of Minkowski cells which partition $P$ and intersect properly as Minkowski sums. Cell $R$ is *mixed*, in particular *$i$-mixed* or *$v_i$-mixed*, if it is the Minkowski sum of $n$ one-dimensional segments $E_j \subset P_j$, which are called edge summands, and one vertex $v_i \in P_i$. Mixed subdivisions having the same mixed cells fall into the same equivalence class called *mixed cell configuration.*

Note that mixed subdivisions contain *faces* of all dimensions between 0 and $n$, the maximum dimension corresponding to cells. Every face of a mixed subdivision of $P$ has a unique description as Minkowski sum of subpolytopes of the $P_i$'s. A mixed subdivision is called *regular* if it is obtained as the projection of the lower hull of the Minkowski sum of lifted polytopes $\{(p_i, \omega_i(p_i)) \mid p_i \in P_i\}$. If the lifting function $\omega := \{\omega_i \ldots, \omega_n\}$ is sufficiently generic, then the induced mixed subdivision is called *tight*, and $\sum_{i=0}^n \dim B_i = \dim \sum_{i=0}^n B_i$, for every cell $\sum_{i=0}^n B_i$.

A monomial of the sparse resultant is called *extreme* if its exponent vector corresponds to a vertex of the Newton polytope $\mathcal{N}(\mathcal{R})$ of the resultant. Let $\omega$ be a sufficiently generic lifting function. The $\omega$-extreme monomial of $\mathcal{R}$ is the monomial with exponent vector that maximizes the inner product with $\omega$; it corresponds to a vertex of $\mathcal{N}(\mathcal{R})$ with outer normal vector $\omega$.

**Proposition 2.** [Stu94]. *For every sufficiently generic lifting function $\omega$, we obtain the $\omega$-extreme monomial of $\mathcal{R}$, of the form*

$$\pm \prod_{i=0}^{n} \prod_{R} c_{i,v_i}^{\mathrm{Vol}(R)}, \qquad (2)$$

*where* $\mathrm{Vol}(R)$ *is the Euclidean volume of $R$, the second product is over all $v_i$-mixed cells $R$ of the regular tight mixed subdivision of $P$ induced by $\omega$, and $c_{i,v_i}$ is the coefficient of the monomial of $f_i$ corresponding to vertex $v_i$.*

**Corollary 3.** *There exists a surjection from the mixed cell configurations onto the set of extreme monomials of the sparse resultant.*

Given supports $A_0, \ldots, A_n$, the Cayley embedding $\kappa$ introduces a new point set

$$C := \kappa\,(A_0, A_1, \ldots, A_n) = \bigcup_{i=0}^{n} (A_i \times \{e_i\}) \subset \mathbb{R}^{2n},$$

where $e_i$ are an affine basis of $\mathbb{R}^n$.

**Proposition 4.** [The Cayley Trick] [MV99, San05]. *There exists a bijection between the regular tight mixed subdivisions of the Minkowski sum $P$ and the regular triangulations of $C$.*

Let $h_0, \ldots, h_n \in \mathbb{C}[t_1, \ldots, t_r]$ be polynomials in parameters $t_i$. The implicitization problem is to compute the prime ideal $I$ of all polynomials $\phi \in \mathbb{C}[x_0, \ldots, x_n]$ which satisfy $\phi(h_0, \ldots, h_n) \equiv 0$ in $\mathbb{C}[t_1, \ldots, t_r]$. We are interested in parametric curves where $r = n = 1$, and generalize $h_i$ to be rational expressions in $\mathbb{C}(t)$. Then $I = \langle \phi \rangle$ is a principal ideal. Note that $\phi \in \mathbb{C}[x_0, x_1]$ is uniquely defined up to sign. The $x_i$ are called implicit variables, $A(\phi)$ is the *implicit support* and $\mathcal{N}(\phi)$ is the implicit polygon. Usually a rational parameterization of a plane curve may be defined by

$$x_i = \frac{P_i(t)}{Q_i(t)},\ i = 0, 1,\ \gcd(P_i(t), Q_i(t)) = 1, \tag{3}$$

where the denominators may be equal. All polynomials have fixed supports. We assume that the parameterization is *proper* i.e. the *degree* of the induced rational map equals 1. This avoids, e.g., having all terms in $t^a$ for some $a > 1$. This assumption is justified by the fact that every rational plane curve has a proper parameterization and there are algorithms for computing it (see [Sed86]).

Define $f_0 = x_0 Q_0(t) - P_0(t), f_1 = x_1 Q_1(t) - P_1(t) \in \mathbb{C}[t]$. Then the following proposition gives an explicit formula for the implicit equation of the parametric curve in terms of a Sylvester resultant.

**Proposition 5.** [CSC98] *Let $f_0, f_1 \in \mathbb{C}[t]$ be non-zero univariate polynomials as above. Then*

$$Res_t(f_0(t), f_1(t)) = c \cdot \phi(x_0, x_1)^q,\ c \in \mathbb{C},$$

*where $q$ is the degree of the parameterization.*

Therefore, $\mathcal{N}(\phi)$ is the Newton polytope of a specialized resultant. Furthermore, since the parameterization is proper, then [SW01]

$$deg_{x_i}(\phi(x_0, x_1)) = \max\{deg_t(P_j(t)), deg_t(Q_j(t))\},\ \{i, j\} = \{0, 1\}.$$

The implicit support predicted by degree bounds are usually non-optimal.

## 3. Rational parameterizations with different denominators

We now turn to rational curves with different denominators. Let

$$f_0(t) = xQ_0(t) - P_0(t),\ f_1(t) = yQ_1(t) - P_1(t)\ \in (\mathbb{C}[x, y])[t],\ \gcd(P_i, Q_i) = 1,$$

where all polynomials have fixed supports and generic coefficients with respect to these supports. Let $c_{ij}$ $(0 \leq j \leq m_i)$, $q_{ij}$ $(0 \leq j \leq k_i)$ denote the coefficients of polynomials $P_i(\mathrm{t})$ and $Q_i(t)$, and $N_i = \mathcal{A}(P_i)$, $D_i = \mathcal{A}(Q_i)$ their supports respectively; note that for $i = 1, 2$, $N_i \neq \emptyset$ and $D_i \neq \emptyset$. Then, the supports of $f_0, f_1$ are

$$A_0 = N_0 \cup D_0 = \{0, a_{01}, \ldots, a_{0n}\} \quad \text{and} \quad A_1 = N_1 \cup D_1 = \{0, a_{11}, \ldots, a_{1m}\},$$

where the $a_{0i}$ and $a_{1j}$ are sorted in ascending order; $a_{00} = a_{10} = 0$ because $\gcd(P_i, Q_i) = 1$. Elements of $A_0, A_1$ are embedded by the Cayley embedding $\kappa$ in $\mathbb{R}^2$. The embedded points are denoted by $(a_{0i}, 0), (a_{1i}, 1)$; by abusing notation, we shall omit the second coordinate.

Recall that each $p \in A_0$ corresponds to a monomial of $f_0$. The corresponding coefficient either lies in $\mathbb{C}$, or is a monomial $q_{0i}x$, or a binomial $q_{0i}x + c_{0j}$, where $q_{0i}, c_{0j} \in \mathbb{C}$. The resultant $\mathcal{R}(f_0, f_1)$ is a polynomial in $x, y, c_{ij}, q_{ij}$. We consider the specialization of coefficients $c_{ij}, q_{ij}$ in order to study $\phi$; this specialization yields the implicit equation. The relevant terms are products of one polynomial in $x$ and one in $y$. The former is the product of powers of terms of the form $q_{0i}x$ or $q_{0i}x + c_{0j}$ ; the $y$-polynomial is obtained analogously. The exponents in $A_0$ and $A_1$ relevant to the implicit polygon are the ones corresponding to coefficients which are non-constant polynomials in $x$. These exponents fall into two different categories: the exponents in $D_0$ and those in $D_0 \setminus N_0$; the latter contains the exponents corresponding to coefficients which are monomials in $x$. An analogous description holds for the second polynomial.

We need consider only $i$-mixed cells associated with a vertex coming from $D_i$ or $D_i \setminus N_i$. For any triangulation, these mixed cells correspond either to triangles with vertices $\{a_{0i}, a_{1\ell}, a_{1r}\}$, where $\ell, r \in \{0, \ldots, m\}$, or to $\{a_{0\ell}, a_{0r}, a_{1j}\}$, where $\ell, r \in \{0, \ldots, n\}$. Given a triangulation, we set

$$e_0 = \sum_{i,\ell,r} \mathrm{Vol}(a_{0i}, a_{1\ell}, a_{1r}), \qquad e_1 = \sum_{\ell,r,j} \mathrm{Vol}(a_{0\ell}, a_{0r}, a_{1j}), \tag{4}$$

where $i, j$ range over all elements of $D_0$ or $D_0 \setminus N_0$ and $D_1$ or $D_1 \setminus N_1$, respectively, and we sum up the normalized volumes of mixed triangles.

In the following, we use the upper (lower, resp.) hull of a convex polygon in $\mathbb{R}^2$ w.r.t. some direction $v \in \mathbb{R}^2$. Let us consider the unbounded convex polygons defined by the computed upper and lower hulls. The union of these two unbounded polygons is the implicit Newton polygon.

**Lemma 6.** *Consider all points $(e_0, e_1)$, defined by expressions (4), over all possible triangulations. The polygon defined by the upper hull of points $(e_0, e_1)$ w.r.t. to vector $(0, 1)$, where the corresponding vertex comes from $D_i$, $i = 0, 1$, and the lower hull of points $(e_0, e_1)$ w.r.t. to vector $(0, 1)$, where the corresponding vertex comes from $D_i \setminus N_i$, $i = 0, 1$, equals the implicit polygon $\mathcal{N}(\phi)$.*

*Proof.* Consider the extreme terms of the resultant, given by theorem 2. After the specialization of the coefficients, those associated with $i$-mixed cells having a vertex $p \in N_i \setminus D_i$ contribute only a coefficient in $\mathbb{C}$ to the corresponding term of $\phi$. This is why they are not taken into account in (4).

Now consider triangles with vertices from $D_i$. By maximizing $e_0$ or $e_1$, as defined in (4), it is clear that we shall obtain the maximum possible exponents in the terms which are polynomials in $x$ and $y$ respectively, hence the largest degrees in $x, y$ in $\phi$. Under certain genericity assumptions, we shall obtain all vertices in the implicit polygon, which appear in its upper hull with respect to vector $(0, 1)$.

Triangles with vertices from $D_i \setminus N_i$ minimize the powers of coefficients corresponding to monomials in the implicit variables. All other coefficients are in $\mathbb{C}$ or are binomials in $x$ (or $y$), so they contain a constant term, hence their product will contain a constant, assuming generic coefficients in the parametric equations. Therefore these are vertices on the lower hull with respect to $(0, 1)$. □

### 3.1. The implicit vertices

For any $p \in A_i,\ i = 0, 1$, let $\mathcal{X}_{D_i}(p)$ and $\mathcal{X}_{D_i \setminus N_i}(p)$ be the characteristic functions of the sets $D_i$ and $D_i \setminus N_i$: $\mathcal{X}_{D_i}(p) = 1$ if $p \in D_i$, and $\mathcal{X}_{D_i}(p) = 0$ otherwise; similarly, $\mathcal{X}_{D_i \setminus N_i}(p) = 1$ if $p \in D_i \setminus N_i$, and $\mathcal{X}_{D_i \setminus N_i}(p) = 0$ otherwise.

We give formulas for the vertex coordinates of $\mathcal{N}(\phi)$. These vertices are not necessarily distinct, and lie on lines $e_0 = 0,\ e_0 = a_{1m},\ e_1 = 0$ and $e_1 = a_{0n}$.

**Theorem 7.**
*(i) The maximum exponent of $x$ in the implicit equation is $e_0^{max} = a_{1m}$. When this is attained, the maximum exponent of $y$ is*

$$e_1^{max}|_{e_0^{max}} = \max(D_0) - \min(D_0) + \mathcal{X}_{D_1}(0) \cdot \min(D_0) + \mathcal{X}_{D_1}(a_{1m}) \cdot (a_{0n} - \max(D_0)),$$

*and the minimum exponent of $y$ is*

$$e_1^{min}|_{e_0^{max}} = \mathcal{X}_{D_1 \setminus N_1}(0) \cdot \min(D_0 \setminus N_0) + \mathcal{X}_{D_1 \setminus N_1}(a_{1m}) \cdot (a_{0n} - \max(D_0 \setminus N_0)).$$

*(ii) The maximum exponent of $y$ in the implicit equation is $e_1^{max} = a_{0n}$. When this is attained, the maximum exponent of $x$ is*

$$e_0^{max}|_{e_1^{max}} = \max(D_1) - \min(D_1) + \mathcal{X}_{D_0}(0) \cdot \min(D_1) + \mathcal{X}_{D_0}(a_{0n}) \cdot (a_{1m} - \max(D_1)),$$

*and the minimum exponent of $x$ is*

$$e_0^{min}|_{e_1^{max}} = \mathcal{X}_{D_0}(0) \cdot \min(D_1) + \mathcal{X}_{D_0}(a_{0n}) \cdot (a_{1m} - \max(D_1)) + \prod_{j \geq 0} \mathcal{X}_{D_0}(a_{0j}) \cdot (\max(D_1) - \min(D_1)).$$

*(iii) The minimum exponent of $x$ in the implicit equation is $e_0^{min} = 0$. When this is attained, the maximum exponent of $y$ is*

$$e_1^{max}|_{e_0^{min}} = \max(N_0 \setminus D_0) - \min(N_0 \setminus D_0) + \mathcal{X}_{D_1}(0) \cdot \min(N_0 \setminus D_0) + \mathcal{X}_{D_1}(a_{1m}) \cdot (a_{0n} - \max(N_0 \setminus D_0)),$$

*and the minimum exponent of $y$ is*

$$e_1^{min}|_{e_0^{min}} = \mathcal{X}_{D_1 \setminus N_1}(0) \cdot \min(N_0) + \mathcal{X}_{D_1 \setminus N_1}(a_{1m}) \cdot (a_{0n} - \max(N_0)).$$

*(iv) The minimum exponent of $y$ in the implicit equation is $e_1^{min} = 0$. When this is attained, the maximum exponent of $x$ is*

$$\begin{aligned} e_0^{max}|_{e_1^{min}} &= \max(N_1) - \min(N_1) + \mathcal{X}_{D_0 \setminus N_0}(0) \cdot \min(N_1) + \\ &\qquad \mathcal{X}_{D_0 \setminus N_0}(a_{0n}) \cdot (a_{1m} - \max(N_1)), \end{aligned}$$

*and the minimum exponent of $x$ is*

$$e_0^{min}|_{e_1^{min}} = \mathcal{X}_{D_0 \setminus N_0}(0) \cdot \min(N_1) + \mathcal{X}_{D_0 \setminus N_0}(a_{0n}) \cdot (a_{1m} - \max(N_1)).$$

*Proof.* We shall prove only case (i), the rest are either symmetric, or similar.

Since the vertex corresponding to the maximum exponent of $y$ when the maximum exponent of $x$ is attained belongs to the upper hull of the implicit polygon, the exponents are obtained by mixed triangles in eq. (4) where $i, j$ range over all elements of $D_0$, $D_1$, respectively. The maximum possible exponent of $x$ is $a_{1m}$, and this is attained by a triangulation in which the entire segment $[0, a_{1m}]$ is visible by any element of $D_0$; recall $D_0 \neq \emptyset$. Then, the maximum exponent of $y$ is attained from any triangulation such that a maximum part of segment $[0, a_{0n}]$ is visible from some points in $D_1$. A triangulation achieving the maximum exponent of $y$ given in the theorem is shown in Figure 2, left subfigure (note that $a_{1i}$ may coincide with 0 or $a_{1m}$); recall $D_1 \neq \emptyset$.

We will show that this exponent of $y$ is the maximum that can be achieved. Since all points to the left of $\min(D_0)$ do not contribute to the exponent of $x$ in eq. (4), any triangulation obtaining the maximum exponent of $x$ (i.e., $a_{1m}$) cannot contain edges connecting these points to points in $A_1$. Then, $0 \in A_1$ is connected to a point in $A_0$ which should belong to $D_0$; if $0 \notin D_1$, 0 should be adjacent to $\min(D_0)$, to minimize that part of segment $[0, a_{0n}]$ visible by 0. A similar argument holds for $a_{1m}$; note that no point to the right of $\max(D_0)$ contributes to the exponent of $x$ in eq. (4).

The vertex corresponding to the minimum exponent of $y$ when the maximum exponent of $x$ is attained belongs to the lower hull of the implicit polygon, hence the exponents are obtained by mixed triangles in eq. (4) where $i, j$ range over all elements of $D_0 \setminus N_0$, $D_1 \setminus N_1$, respectively. Then, the minimum exponent of $y$ stated in the theorem can be achieved by the triangulations shown in Figure 2; the center figure corresponds to the case that $a_{1m} \in N_1$, the right to the case that $a_{1m} \in D_1 \setminus N_1$. As above, in order to attain the maximum exponent of $x$, $0 \in A_1$ is connected to a point in $A_0$ which should in fact belong to $D_0 \setminus N_0$; if $0 \in D_1 \setminus N_1$, the minimum is obtained if 0 is connected to $\min D_0 \setminus N_0$; this leads to the first term of the expression of the exponent in the theorem. The second term is obtained by a similar argument for $a_{1m}$. □

Theorem 7 yields a set of 8 (not necessarily distinct) possible vertices for $\mathcal{N}(\phi)$. Consider the rectangle $ABCD$, with vertices defined by the intersections of lines $e_0 = 0$, $e_0 = a_{1m}$, $e_1 = 0$, and $e_1 = a_{0n}$; in particular, $A = (0, 0)$, $B = (0, a_{0n})$, $C = (a_{1m}, a_{0n})$, and $D = (a_{1m}, 0)$. $\mathcal{N}(\phi)$ is defined from this rectangle after an appropriate number of corner cuts.

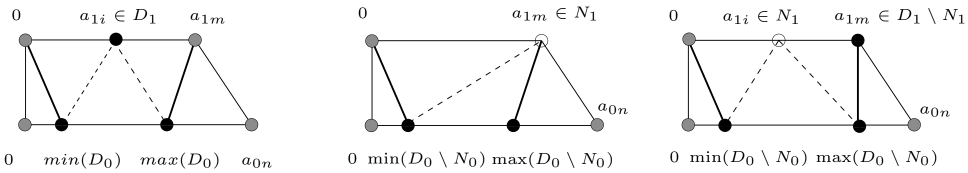


FIGURE 2. The triangulations of $C$ in thm. 7 giving vertices $e_1^{max}|_{e_0^{max}}$ and $e_1^{min}|_{e_0^{max}}$; the color of the disks (black, grey, white) indicates membership (belongs, does not belong, may belong, respectively) to $D_i$ or $D_i \setminus N_i$.

In order to have a cut, a term $\mathcal{X}_A(t) \cdot r$ in the expression of $e_i^{min}|_\epsilon$ yields the condition "$t \in A$ and $r \neq 0$," whereas the same term in the expression of $e_i^{max}|_\epsilon$ yields the condition "$t \notin A$ and $r \neq 0$." Then, the following follows from theorem 7:

**Corollary 8.** *The conditions for a cut in each of the four corners of $ABCD$ are:*

- cut at $A$: $(0 \in D_0 \backslash N_0$ *and* $0 \in D_1 \backslash N_1)$ *or* $(a_{0n} \in D_0 \backslash N_0$ *and* $a_{1m} \in D_1 \backslash N_1)$;
- cut at $B$: $(0 \in D_0$ *and* $0 \in N_1 \setminus D_1)$ *or* $(a_{0n} \in D_0$ *and* $a_{1m} \in N_1 \setminus D_1)$;
- cut at $C$: $(0 \in N_0 \backslash D_0$ *and* $0 \in N_1 \backslash D_1)$ *or* $(a_{0n} \in N_0 \backslash D_0$ *and* $a_{1m} \in N_1 \backslash D_1)$;
- cut at $D$: $(0 \in N_0$ *and* $0 \in D_1 \setminus N_1)$ *or* $(a_{0n} \in N_0$ *and* $a_{1m} \in D_1 \setminus N_1)$.

From this corollary and from the fact that if $D_0 = A_0$, then $e_0^{max}|_{e_1^{max}} = e_0^{max}|_{e_1^{max}} = a_{1m}$, we have:

**Corollary 9.** *There can be at most two corner cuts in different corners of rectangle $ABCD$, defined by the vertices of theorem 7 which do not coincide.*

Suppose there is only one corner cut in rectangle $ABCD$. Then, there may exist an additional vertex of $\mathcal{N}(\phi)$ which does not follow from theorem 7. We define:

$$\delta_A = \det \begin{bmatrix} a_{0n} - \max(N_0) & a_{1m} - \max(N_1) \\ \min(N_0) & \min(N_1) \end{bmatrix}, \tag{5}$$

$$\delta_B = \det \begin{bmatrix} a_{0n} - \max(N_0 \setminus D_0) & a_{1m} - \max(D_1) \\ \min(N_0 \setminus D_0) & \min(D_1) \end{bmatrix}. \tag{6}$$

Moreover, $\delta_C$ is defined by replacing the sets $N_i$ by the sets $D_i$ in equation (5) and $\delta_D$ is defined by replacing the set $N_0 \setminus D_0$ by the set $D_0 \setminus N_0$, and the set $D_1$ by the set $N_1$ in equation (6).

**Theorem 10.** *Suppose the vertices of theorem 7 yield only one corner cut in rectangle $ABCD$. Then, the implicit polygon is equal to the cut rectangle $ABCD$ unless:*

(i) *cut at A:* $0, a_{0n} \in D_0 \setminus N_0$ *and* $0, a_{1m} \in D_1 \setminus N_1$ *and* $\delta_A \neq 0$*, in which case there exists a vertex* $p$ *s.t.*

$$p = (\min(N_1), a_{0n} - \max(N_0)) \text{ if } \delta_A < 0, \text{ and}$$
$$p = (a_{1m} - \max(N_1), \min(N_0)) \text{ if } \delta_A > 0.$$

(ii) *cut at B:* $0, a_{0n} \in D_0$ *and* $0, a_{1m} \in N_1 \setminus D_1$ *and* $\delta_B \neq 0$*, in which case there exists a vertex* $p$ *s.t.*

$$p = (\min(D_1), \max(N_0 \setminus D_0)) \text{ if } \delta_B < 0, \text{ and}$$
$$p = (a_{1m} - \max(D_1), a_{0n} - \min(N_0 \setminus D_0)) \text{ if } \delta_B > 0.$$

(iii) *cut at C:* $0, a_{0n} \in N_0 \setminus D_0$ *and* $0, a_{1m} \in N_1 \setminus D_1$ *and* $\delta_C \neq 0$*, in which case there exists a vertex* $p$ *s.t.*

$$p = (a_{1m} - \min(D_1), \max(D_0)) \text{ if } \delta_C < 0, \text{ and}$$
$$p = (\max(D_1), a_{0n} - \min(D_0)) \text{ if } \delta_C > 0.$$

(iv) *cut at D:* $0, a_{0n} \in N_0$ *and* $0, a_{1m} \in D_1 \setminus N_1$ *and* $\delta_D \neq 0$*, in which case there exists a vertex* $p$ *s.t.*

$$p = (\min(D_0 \setminus N_0), \max(N_1)) \text{ if } \delta_D < 0, \text{ and}$$
$$p = (a_{0n} - \max(D_0 \setminus N_0), a_{1m} - \min(N_1)) \text{ if } \delta_D > 0.$$

*Proof.* We prove only case (ii). The other cases are either similar or symmetric.

Suppose theorem 7 yields a cut in rectangle $ABCD$, excluding vertex $B$. Then, corollary 8 implies that $0 \in D_0$ and $0 \in N_1 \setminus D_1$, or $a_{0n} \in D_0$ and $a_{1m} \in N_1 \setminus D_1$.

Consider the case in which $0 \in D_0$, $0 \in N_1 \setminus D_1$, and ($a_{0n} \notin D_0$ or $a_{1m} \notin N_1 \setminus D_1$ or both). Then, $e_1^{max}|_{e_0^{min}} = a_{0n} - \min(N_0 \setminus D_0)$ and $e_0^{min}|_{e_1^{max}} = \min(D_1)$ yielding the vertices $(0, a_{0n} - \min(N_0 \setminus D_0))$ and $(\min(D_1), a_{0n})$. Suppose, for contradiction, that there exists a triangulation $T$ corresponding to a point $p_T = (x_T, y_T)$ with $x_T < \min(D_1)$ and $y_T > a_{0n} - \min(N_0 \setminus D_0)$. Consider the edges $a_{0i}$-$a_{1j}$ of $T$; as these edges do not cross, they can be ordered from left to right. The leftmost edge is 0-0 with $0 \in D_0$ and $0 \in N_1 \setminus D_1$. Let $a_{0i}$-$a_{1j}$ be the leftmost edge such that either $a_{0i} \notin D_0$ or $a_{1j} \notin N_1 \setminus D_1$; exactly one of these two conditions will hold, since any two consecutive such edges share an endpoint. If $a_{0i} \notin D_0$, then all the points $0, \ldots, a_{1j} \in N_1 \setminus D_1$, and thus no portion of the segment $[0, a_{0i}]$ contributes to the $y$-coordinate $y_t$ of $p_T$, i.e., $y_T \leq a_{0n} - a_{0i} \leq a_{0n} - \min(N_0 \setminus D_0)$, a contradiction. Similarly, if $a_{1j} \notin N_1 \setminus D_1$, that is, $a_{1j} \in D_1$, then all the points $0, \ldots, a_{0i} \in D_0$, and thus the entire segment $[0, a_{1j}]$ contributes to the $x$-coordinate $x_t$, i.e., $x_T \geq a_{1j} \geq \min(D_1)$, a contradiction again. Therefore, the cut in the rectangle $ABCD$ that excludes vertex $B$ is the only possible one and the implicit polygon equals the polygon defined by the rectangle and the corner cut. The case in which $a_{0n} \in D_0$, $a_{1m} \in N_1 \setminus D_1$, and ($0 \notin D_0$ or $0 \notin N_1 \setminus D_1$ or both) is right-to-left symmetric yielding a similar result.

Finally, we consider the case in which $0, a_{0n} \in D_0$ and $0, a_{1m} \in N_1 \setminus D_1$. Then, $e_1^{max}|_{e_0^{min}} = \max(N_0 \setminus D_0) - \min(N_0 \setminus D_0)$ and $e_0^{min}|_{e_1^{max}} = \min(D_1) + a_{1m} - \max(D_1)$ leading to points $q_1 = (0, \max(N_0 \setminus D_0) - \min(N_0 \setminus D_0))$ and $q_2 = (\min(D_1) + a_{1m} - \max(D_1), a_{0n})$. Consider the points $p_1 = (\min(D_1), \max(N_0 \setminus D_0))$ and $p_2 = (a_{1m} - \max(D_1), a_{0n} - \min(N_0 \setminus D_0))$. It is not difficult to see that one can obtain triangulations corresponding to these points. Points $q_1$, $q_2$, $p_1, p_2$ form a parallelogram which degenerates to a line segment if $\delta_B = 0$; otherwise, $p_1$ ($p_2$, resp.) is above the line through $q_1, q_2$ if $\delta_B < 0$ ($\delta_B > 0$, resp.).

W.l.o.g. assume $\delta_B < 0$. We will show that $q_1p_1$ is an edge of $\mathcal{N}(\phi)$; suppose, for contradiction, that there exists a triangulation $T$ corresponding to a point $p_T = (x_T, y_T)$ which has $x_T < \min(D_1)$, $y_T > \max(N_0 \setminus D_0) - \min(N_0 \setminus D_0)$ and lies above the line through $q_1, p_1$. Since $0 \in D_0$ and $0 \in N_1 \setminus D_1$, we consider the ordered edges $a_{0i}$-$a_{1j}$ of $T$ (from left to right) and as above we show that either the entire segment $[0, \min(D_1)]$ contributes to the $x$-coordinate of $p_T$ or no part of the segment $[0, \min(N_0 \setminus D_0)]$ contributes to its $y$-coordinate; the former is in contradiction with the fact that $x_T < \min(D_1)$, and thus the latter case holds. Moreover, by considering the edges $a_{0i}$-$a_{1j}$ of $T$ from right to left, we can show that either the entire segment $[\max(D_1), a_{1m}]$ contributes to the $x$-coordinate of $p_T$ or no part of the segment $[\max(N_0 \setminus D_0), a_{0n}]$ contributes to its $y$-coordinate; the latter case, in conjunction with the latter case of the previous observation, is in contradiction with $y_T > \max(N_0 \setminus D_0) - \min(N_0 \setminus D_0)$, and hence the former case holds. Thus, $x_T \geq a_{1m} - \max(D_1)$ and $y_T \leq a_{0n} - \min(N_0 \setminus D_0)$. For $p_T$ to be above the line through $q_1$ and $p_1$, it should hold that

$$\frac{y_T - (\max(N_0 \setminus D_0) - \min(N_0 \setminus D_0))}{x_T} > \frac{\min(N_0 \setminus D_0)}{\min(D_1)};$$

this is not possible because

$$\frac{y_T - (\max(N_0 \setminus D_0) - \min(N_0 \setminus D_0))}{x_T} \leq \frac{a_{0n} - \max(N_0 \setminus D_0)}{a_{1m} - \max(D_1)}$$

$$\text{and} \quad \delta_B < 0 \quad \Longrightarrow \quad \frac{a_{0n} - \max(N_0 \setminus D_0)}{a_{1m} - \max(D_1)} < \frac{\min(N_0 \setminus D_0)}{\min(D_1)}.$$

Therefore, the segment $q_1p_1$ is an edge of $\mathcal{N}(\phi)$. For $\delta_B < 0$, in a similar fashion we can show that the segment $q_2p_1$ is also an edge of $\mathcal{N}(\phi)$. The cases for $\delta_B > 0$ are symmetric involving point $p_2$. □

**Example 2.**

$$x = \frac{a + t^2}{ct}, \; y = \frac{b}{dt}, \quad a, b, c, d \neq 0.$$

With generic coefficients, the denominators are different. The input supports are $N_0 = \{0, 2\}$, $N_1 = \{0\}$, $D_0 = D_1 = \{1\}$. Theorem 7 yields points $(1, 1)$, $(1, 1)$, $(0, 2), (0, 2), (0, 2), (0, 0), (0, 0), (0, 0)$, in the order stated by the theorem, which define the actual implicit polygon since $\phi = ad^2y^2 - bcdxy + b^2$ .

**Example 3.**

$$x = \frac{t^7+t^4+t^3+t^2}{t^3+1},\ y = \frac{t^5+t^4+t}{t^5+t^2+1}.$$

The input supports are $N_0 = \{2,3,4,7\},\ N_1 = \{1,4,5\},\ D_0 = \{0,3\}$ and $D_1 = \{0,2,5\}$. Theorem 7 yields points $(5,7),(5,0),(5,7),(0,7),(0,7),(0,2),(5,0),(1,0)$ in the order stated by the theorem. These points define the actual implicit polygon.

**Example 4.** For the unit circle, $x = 2t/(t^2+1),\ y = (1-t^2)/(t^2+1)$, the supports are $N_0 = \{1\},\ D_0 = \{0,2\}$, and $N_1 = D_1 = \{0,2\}$. The set $C = \kappa(A_0, A_1)$ has 5 triangulations shown in figure 3 which, after applying proposition 2, give the terms $y^2-1,\ x^2y^2-2x^2y+x^2$ and $x^2y^2+2x^2y+x^2$. This method yields points $(2,2),(2,0),(0,2),(0,0)$. By degree bounds, we end up with vertices $(2,0),(0,2),(0,0)$. Interestingly, to see the cancellation of term $x^2y^2$ it does not suffice to consider only terms coming from extremal monomials in the resultant. See example 8 for a treatment taking into account the identical denominators.

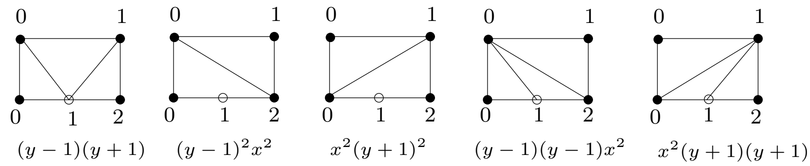


Figure 3. The triangulations of $C$ in example 4, and the corresponding terms.

**Example 5.** Consider the parameterization

$$x = \frac{t^3+2t^2+t}{t^2+3t-2},\ y = \frac{t^3-t^2}{t-2}.$$

The supports are $N_0 = \{1,2,3\},\ D_0 = \{0,1,2\}$, and $N_1 = \{2,3\}, D_1 = \{0,1\}$. Theorem 7 yields points $(3,2),(3,0),(1,3),(0,3),(0,3),(0,1),(3,0),(2,0)$, in the order stated by the theorem, which define the actual implicit polygon. The implicit polygon is shown in figure 1, right.

## 4. Rational parameterizations with equal denominators

We study rationally parameterized curves, when both denominators are same.

$$x = \frac{P_0(t)}{Q(t)},\ y = \frac{P_1(t)}{Q(t)},\ \gcd(P_i(t), Q(t)) = 1,\ P_i, Q \in \mathbb{C}[t],\ i = 0, 1, \tag{7}$$

where the $P_i, Q$ have fixed supports and generic coefficients. If some $P_i(t), Q(t)$ have a nontrivial GCD, then common terms are divided out and the problem reduces to the case of different denominators. In general, the $P_i, Q$ are Laurent polynomials, but this case can be reduced to the case of polynomials by shifting the supports.

This section is also useful if the two parametric expressions have the same numerator and different denominators. Then, we consider implicit variables $x^{-1}, y^{-1}$, compute the implicit polygon, and transform it to get the implicit polygon of the original problem. Similarly, if the numerator of one parametric expression equals the denominator of the other, then we can again apply the tools of this section.

Considering the more general case of different denominators does not lead to optimal implicit support, because this does not exploit the fact that the coefficients of $Q(t)$ are the same in the polynomials $xQ-P_0, yQ-P_1$. Therefore, we introduce a new variable $r$ and consider the following system

$$f_0 = xr - P_0(t),\ f_1 = yr - P_1(t),\ f_2 = r - Q(t) \in \mathbb{C}[t, r]. \tag{8}$$

By eliminating $t, r$ the resultant gives, for generic coefficients, the implicit equation in $x, y$. Consider the parameterization

$$\tau : \mathbb{P} \to \mathbb{P}^2 : (t : t_0) \mapsto (x_0 : x_1 : x_2) = (P_0^h : P_1^h : Q^h), \tag{9}$$

where $P_0^h, P_1^h, Q^h$ are the homogenizations of $P_0, P_1, Q$. The resultant of polynomials defined by equations (9) is homogeneous in $x_0, x_1, x_2$ and generically equals the implicit equation $\Phi \in \mathbb{C}[x_0, x_1, x_2]$ of parameterization $\tau$. The resultant of polynomials (8) is the de-homogenization of $\Phi$. Let the input supports be

$$B_i = \mathcal{A}(P_i), i = 0, 1,\ B_2 = \mathcal{A}(Q),\ \text{where } B_i = \{b_{iL}, \ldots, b_{iR}\}, i = 0, 1, 2,$$

where indices $L, R$ denote the leftmost and rightmost points respectively, i.e., $b_{iL},\ b_{iR}$ are the minimum and maximum points respectively in $B_i$. The supports of the $f_i$ are

$$A_0 = \{a_{00}, a_{0L}, \ldots, a_{0R}\}, A_1 = \{a_{10}, a_{1L}, \ldots, a_{1R}\}, A_2 = \{a_{20}, a_{2L}, \ldots, a_{2R}\} \in \mathbb{N}^2,$$

where

- each point $a_{i0} = (0, 1)$, for $i = 0, 1, 2$, corresponds to the unique term in $f_i$ which depends on $r$,
- each other point $a_{it}$, for $t \neq 0$, is of the form $(b_{it}, 0)$, for one $b_{it} \in B_i$.

One could think that index $L = 1$ whereas each $R$ equals the cardinality of the respective $B_i$. By the above hypotheses $A_2$ or both $A_0, A_1$ contain $(0, 0)$.

**Lemma 11.** *$MV_{\mathbb{Z}}(B_i \cup B_j) = MV_{\mathbb{Z}^2}(A_i, A_j),\ i, j \in \{0, 1, 2\}$, where $MV_{\mathbb{Z}^d}$ denotes mixed volume in $\mathbb{Z}^d$.*

*Proof.* Let $\mathrm{CH}(B_i) = [m_i, l_i]$, $\mathrm{CH}(B_j) = [m_j, l_j]$ be intervals in $\mathbb{N}$. If $m_i \leq m_j$ and $l_i \leq l_j$, then $\mathrm{MV}_{\mathbb{Z}}(B_i \cup B_j) = l_j - m_i$. Take a mixed subdivision of $A_i + A_j$, with unique mixed cell $((0, 1), (m_i, 0)) + ((0, 1), (l_j, 0))$, hence $\mathrm{MV}_{\mathbb{Z}^2}(A_i, A_j) = l_j - m_i$. If $m_i \leq m_j \leq l_j \leq l_i$, then $\mathrm{MV}_{\mathbb{Z}}(B_i \cup B_j) = l_i - m_i$, and a similar subdivision as above yields a unique mixed cell with this volume. The other cases are symmetric. □

In what follows, we shall make use of integer $u = \max\{b_{0R}, b_{1R}, b_{2R}\}$.

Let $C_i = \mathrm{CH}(A_i)$ and consider the mixed subdivisions of $C = C_0 + C_1 + C_2$. The following points lie on the boundary of $C$: $(u, 2), (0, 3), (0, 2), (b_{0L}+b_{1L}+b_{2L}, 0)$ and $(b_{0R} + b_{1R} + b_{2R}, 0)$.

The vertices $e_0, e_1, e_2$ of implicit Newton polytope $\mathcal{N}(\Phi)$ correspond to monomials in $x_0, x_1, x_2$; the power of each $x_i$ is determined by the volumes of $a_{i0}$-mixed (or simply $i$-mixed) cells, for $i = 0, 1, 2$. This leads us to computing mixed subdivisions of three polygons in the plane.

**Lemma 12.** [Cell types] *In any mixed subdivision of $C$, the $i$-mixed cells, with vertex summand $a_{i0}$, for some $i \in \{0,1,2\}$, have an edge summand $(a_{j0}, a_{jh})$, $i \neq j$, $h > 0$. Their second edge summand is from $B_l$, where $\{i,j,l\} = \{0,1,2\}$ and classifies the $i$-mixed cells in two types:*
*(I) If it is $(a_{l0}, a_{lm})$, where $a_{lm} = (b_{lm}, 0)$, then the cell vertices are $(0,3)$, $(b_{jh}, 2)$, $(b_{lm}, 2)$, $(b_{jh} + b_{lm}, 1)$, provided $b_{jh} \neq b_{lm}$.*
*(II) If it is $(a_{lt}, a_{lm})$, where $a_{lt} = (b_{lt}, 0), a_{lm} = (b_{lm}, 0)$, then the cell vertices are $(b_{lt}, 2), (b_{lm}, 2)$, $(b_{jh} + b_{lt}, 1), (b_{jh} + b_{lm}, 1)$.*

*Proof.* Any mixed cell has two non-parallel edge summands, hence one of the edges is $(a_{j0}, a_{jh})$ for some $i \neq j$, $h > 0$. The rest of the statements follow from the definition of a mixed subdivision. □

Observe that for every type-II cell, there is a non-mixed cell with vertices $(0,3)$, $(b_{lt}, 2), (b_{lm}, 2)$.

**Example 6.** We consider the folium of Descartes:

$$x = \frac{3t^2}{1+t^3}, y = \frac{3t}{1+t^3} \Rightarrow \phi = x^3 + y^3 - 3xy = 0.$$

Now $f_0 = xr - 3t^2, f_1 = yr - 3t, f_2 = r - (t^3 + 1)$. Figure 4 shows the Newton polygons, $C$ and two mixed subdivisions. The shaded triangle is the only unmixed cell with nonzero area; it is a copy of $C_2$. The first subdivision shows two cells of type I, of area 1 and 2, which yield factors $x$ and $y^2$ respectively, to give term $xy^2$. The second subdivision has one cell of type II and area 3, which yields term $x^3$. We shall obtain an optimal support in example 9. Now, $u = 3$ which equals the total degree of $\phi$.

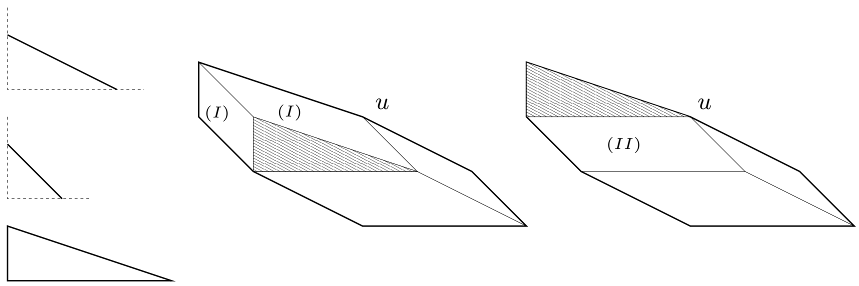


FIGURE 4. Example 6: polygons $C_i$, and two mixed subdivisions of $C$.

Consider segment $E$ defined by vertices $(0,2), (u,2)$ in $C$.

**Lemma 13.** *The resultant of the $f_i$'s $\in \mathbb{C}[t, r]$ defined by equations (8) is homogeneous, of degree $u$, w.r.t. the coefficients of the $a_{i0}$, for $i = 0, 1, 2$.*

*Proof.* Consider any mixed subdivision of $C$ and the cells of type I and II. Consider these cells as closed polygons: We claim that their union contains segment $E$. Then, it is easy to see that the total volume of these cells equals $u$.

Consider the closed cells that intersect $E$. If the intersection lies in the cell interior, then it is a parallelogram, hence is mixed with vertex summand $(0, 1)$, thus it is of type I. If the intersection is a cell edge, say $(a_{kl}, a_{km})$, for $k \in \{0, 1, 2\}$ and $1 \leq l < m$, then the cell above $E$ is unmixed, namely a triangle with basis $(a_{kl}, a_{km})$ and apex at $(0, 3)$. In this case, the cell below $E$ is mixed of type II. □

Generically, $u$ equals the total degree of every term in the implicit equation $\phi(x, y)$ w.r.t. $x, y$ and the coefficient of $r$ in $f_2$. The degree of $\Phi(x_0, x_1, x_2)$ is $u$.

In the following, we focus on segment $E$ and subsegments defined by points $(b_{it}, 2) \in L, i \in \{0, 1, 2\}$. Usually, we shall omit the ordinate, so the corresponding segments will be denoted by $[b_{jt}, b_{kl}]$. We say that such a segment contributes to some coordinate $e_i$ when a $i$-mixed cell of the mixed subdivision contains this segment. Moreover,

- a type-I, $i$-mixed cell $a_{i0} + (a_{j0}, a_{jt}) + (a_{k0}, a_{kl})$ is identified with segment $[b_{jt}, b_{kl}]$.
- a type-II, $i$-mixed cell $a_{i0} + (a_{jt}, a_{js}) + (a_{k0}, a_{kl})$ is identified with segment $[b_{jt}, b_{js}]$ and the coordinate $e_i$ to which it contributes.

We show that one needs to examine only subsegments defined by endpoints $b_{iL}, b_{iR} \in B_i$. This is equivalent to saying that it suffices to consider mixed subdivisions induced by linear liftings.

**Theorem 14.** *Let $S$ be a mixed subdivision of $C_0 + C_1 + C_2$, where an internal point $b_i \in B_i$ defines a 0-dimensional face $(b_i, 2) = (b_i, 0) + (0, 1) + (0, 1) \in L$. Then, the point of $\mathcal{N}(\phi)$ obtained by $S$ cannot be a vertex because it is a convex combination of points obtained by other mixed subdivisions defined by points of $B_0, B_1, B_2$ which are either endpoints, or are used in defining $S$ except from $(b_i, 2)$.*

The theorem is established by lemmas 15, 16 and 17. We shall construct mixed subdivisions that yield points in the $e_k e_j$-plane whose convex hull contains the initial point. All cells of the original subdivision which are not mentioned are taken to be fixed, therefore we can ignore their contribution to $e_k, e_j$. All convex combinations in these lemmas are decided by the $3 \times 3$ orientation determinant; cf expression (11).

**Lemma 15.** [II-II] *Consider the setting of theorem 14 and suppose that $(b_i, 2)$ is a vertex of two adjacent type II cells. Then, the theorem follows.*

*Proof.* If both cells are $j$-mixed, then the same point in $e_k e_j$-plane is obtained by one $j$-mixed cell equal to their union, $\{i, j, k\} = \{0, 1, 2\}$. If the cells are $j$- and $k$-mixed, then there are two mixed subdivisions yielding points in the $e_k e_j$-plane, which define a segment that contains the initial point. The subdivisions have one $j$-mixed or one $k$-mixed cell respectively, intersecting the entire subsegment. □

**Lemma 16.** [I-I] *Consider the setting of theorem* 14 *and suppose that* $(b_i, 2)$ *is a vertex of two adjacent type I cells. W.l.o.g., these are* $k$*- and* $j$*-mixed cells,* $\{i, j, k\} = \{0, 1, 2\}$*. Then, the theorem follows.*

*Proof.* Let $[b_{jl}, b_i], [b_i, b_{kt}]$ be the subsegments defined on $E$ by the two mixed cells, and let $\alpha, \beta$ be their respective lengths. Since $b_i$ is internal, $b_{iR}$ lies to its right-hand side and $b_{iL}$ lies to its left-hand side.

Case $b_{iR} < b_{kt}$ and $b_{iL} > b_{jl}$. Let $\gamma = b_i - b_{iL}$ and $\delta = b_{iR} - b_i$. The initial point $(\alpha, \beta)$ shall be enclosed by two points. The mixed subdivision with type-I cells corresponding to $[b_{jl}, b_{iR}]$ and $[b_{iR}, b_{kt}]$ yields point $(\alpha + \delta, \beta - \delta)$. The subdivision with type-I cells corresponding to $[b_{jl}, b_{iL}], [b_{iL}, b_{kt}]$ yields point $(\alpha - \gamma, \beta + \gamma)$.

Case $b_{iR} < b_{kt}$ and $b_{iL} < b_{jl}$. Let $\gamma = b_{jl} - b_{iL}$ and $\delta = b_{iR} - b_i < \beta$. The initial point is $(\alpha + v_k, \beta + v_j)$, where $v_k, v_j \geq 0$ is the contribution to $e_k, e_j$ respectively from subsegment $[b_{iL}, b_{jl}]$, and $v_k + v_j \leq \gamma$. Now consider 3 mixed subdivisions on $[b_{iL}, b_{kt}]$: The first containing the type-II $k$-mixed cell $[b_{iL}, b_{iR}]$ and the type-I $j$-mixed cell $[b_{iR}, b_{kt}]$ gives $(\alpha + \gamma + \delta, \beta - \delta)$. The second containing the type-I $j$-mixed cell $[b_{iL}, b_{kt}]$ gives $(0, \alpha + \beta + \gamma)$. The third containing the type-I $i$-mixed cell $[b_{jl}, b_{kt}]$ and the initial cells in $[b_{iL}, b_{jl}]$, gives $(v_k, v_j)$.

Case $b_{iR} > b_{kt}$ and $b_{iL} > b_{jl}$. Let $\gamma = b_i - b_{iL} < \alpha$ and $\delta = b_{iR} - b_{kt}$. The initial point is $(\alpha + v_k, \beta + v_j)$, where $v_k, v_j \geq 0$ is the contribution to $e_k, e_j$ respectively from $[b_{kt}, b_{iR}]$, and $v_k + v_j \leq \delta$. Now consider 3 mixed subdivisions on $[b_{jl}, b_{iR}]$: The first containing the type-I $i$-mixed cell $[b_{jl}, b_{kt}]$ and the initial cells in $[b_{kt}, b_{iR}]$, gives point $(v_k, v_j)$. The second containing the type-I $k$-mixed cell $[b_{jl}, b_{iR}]$, gives point $(\alpha + \beta + \delta, 0)$. The third containing the type-I $k$-mixed cell $[b_{jl}, b_{iL}]$ and the type-II $j$-mixed cell $[b_{iL}, b_{iR}]$, gives $(\alpha - \gamma, \beta + \gamma + \delta)$.

Case $b_{iR} > b_{kt}$ and $b_{iL} < b_{jl}$ is analogous to the previous ones. □

**Lemma 17.** [I-II] *Consider the setting of theorem* 14 *and suppose that* $(b_i, 2)$ *is a vertex of two adjacent type II and I cells. Wlog, these are* $k$*- and* $j$*-mixed cells,* $\{i, j, k\} = \{0, 1, 2\}$*. Then, the theorem follows.*

*Proof.* Let $[b_{il}, b_i], [b_i, b_{kt}]$ be the subsegments defined on $E$ by the two mixed cells, and let $\alpha, \beta$ be their respective lengths. Since $b_i$ is internal, $b_{iR}$ lies to its right-hand side. Moreover, the initial $k$-mixed cell implies the existence of 1-dimensional face $(b_i, 2) + a_{k0} + E_{jl}$, for some edge $E_{jl} = (a_{j0}, a_{jl}) \subset B_j$. The initial $j$-mixed cell implies the existence of 1-face $(b_i, 2) + a_{j0} + E_{kt}$, for edge $E_{kt} = (a_{k0}, a_{kt}) \subset B_k$. The second 1-face cannot be to the left of the first, hence $b_{jl} \leq b_{kt}$. So $b_{jL} \leq b_{kt}$.

Case $b_{iR} \leq b_{kt}$. The initial point $(\alpha, \beta)$ shall be enclosed by two points. The mixed subdivision with type-I cell $[b_{il}, b_{kt}]$ yields point $(0, \alpha + \beta)$. The subdivision with type-II and type-I cells corresponding to $[b_{il}, b_{iR}], [b_{iR}, b_{kt}]$ sets $e_k > \alpha, e_j < \beta$, where $e_k + e_j = \alpha + \beta$.

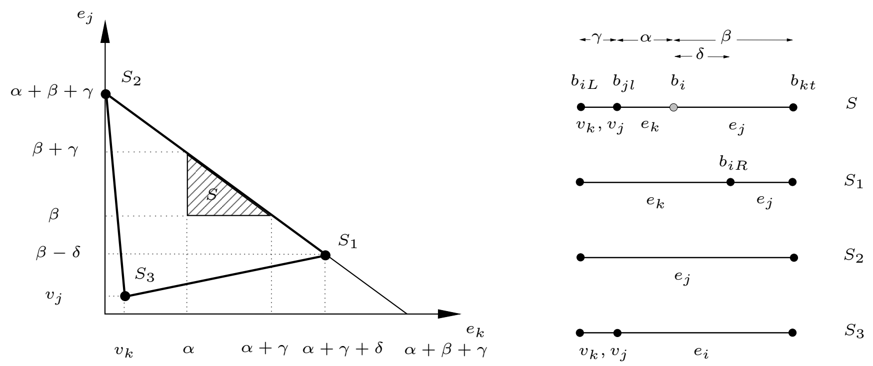


FIGURE 5. The 3 points that enclose the point given by $S$ and the corresponding mixed subdivisions for the second case of Lemma 16.

Case $b_{iR} > b_{kt}$ and $b_{jL} > b_{il}$. Consider subsegment $[b_{il}, b_{iR}]$: the initial point is $(\alpha + v_k, \beta + v_j)$, where $v_k, v_j \geq 0$ is the contribution to $e_k, e_j$ respectively from subsegment $[b_{kt}, b_{iR}]$, and $v_k + v_j \leq \gamma = b_{iR} - b_{kt}$. Now consider 3 mixed subdivisions on $[b_{il}, b_{iR}]$: One $k$-mixed cell $[b_{il}, b_{iR}]$ gives point $(\alpha+\beta+\gamma, 0)$. One $j$-mixed cell $[b_{il}, b_{kt}]$ and the initial cells in $[b_{kt}, b_{iR}]$ give $(v_k, \alpha+\beta+v_j)$. One $k$-mixed cell $[b_{il}, b_{jL}]$, one $i$-mixed cell $[b_{jL}, b_{kt}]$ and the initial cells in $[b_{kt}, b_{iR}]$ give $(e_k + v_k, v_j)$, for some $e_k \leq \alpha + \beta$.

Case $b_{iR} > b_{kt}$ and $b_{jL} \leq b_{il}$ is analogous to the previous ones. □

In the next lemma and corollary, we shall determine certain points in $\mathcal{N}(\Phi)$. We shall later see that among these points lie the vertices of $\mathcal{N}(\Phi)$ and, therefore, from these points we can recover the vertices of $\mathcal{N}(\phi)$. Recall that $\mathrm{MV}_i = \mathrm{MV}_{\mathbb{Z}^2}(A_j, A_k)$, where $\{i,j,k\} = \{0,1,2\}$.

**Lemma 18.** *Given supports $B_0, B_1, B_2$, let $b_{tL} = \min\{b_{iL}, b_{jL}\}$, $b_{mR} = \max\{b_{iR}, b_{jR}\}$ and $\Delta = [b_{tL}, b_{mR}]$, for $i \neq j \in \{0,1,2\}$ and $t, m \in \{i,j\}$ not necessarily distinct. Set $e_\lambda = |\Delta|$, where $\lambda \in \{0,1,2\} \setminus \{i,j\}$, and $e_i = e_j = 0$. Then, add $b_{tL}$ to $e_\tau$, where $\tau \in \{i,j\} \setminus \{t\}$, and add $u - b_{mR}$ to $e_\mu$, where $\mu \in \{i,j\} \setminus \{m\}$. Then, $(e_0, e_1, e_2)$ is a vertex of $\mathcal{N}(\Phi)$.*

*Proof.* Clearly $\Delta = \mathrm{CH}(B_i \cup B_j) \subseteq [0, u]$, so $\mathrm{MV}_\lambda = |\Delta|$. It is possible to construct a mixed subdivision that yields the implicit vertex. If $t \neq m$, then the mixed subdivision contains a type-I mixed cell $(a_{t0}, a_{tL}) + (a_{m0}, a_{mR_m}) + a_{\lambda 0}$ which intersects segment $E$ at subsegment $[b_{tL}, b_{mR}]$. This contributes $\mathrm{MV}_\lambda = b_{mR} - b_{tL}$ to $e_\lambda$. There is a type-I cell $(a_{\lambda 0}, a_{\lambda L}) + (a_{t0}, a_{tL}) + a_{\tau 0}$ which intersects $E$ at subsegment $[0, b_{tL}]$. This contributes $b_{tL}$ to $e_\tau$. Similarly, we assign the area $u - b_{mR}$ of the type-I cell $(a_{\lambda 0}, a_{\lambda R}) + (a_{m0}, a_{mR_m}) + a_{\mu 0}$ to $e_\mu$.

If $t = m$, then $\Delta$ is an edge of one of the initial Newton segments, say $B_t$, and $\Delta = [b_{tL}, b_{tR}]$. The mixed subdivision contains the type-II mixed cell

$(a_{\tau 0}, a_{\tau L}) + (a_{tL}, a_{tR}) + a_{\lambda 0}$ which contributes $\mathrm{MV}_\lambda = |\Delta| = b_{tR} - b_{tL}$ to $e_\lambda$. There are also two type-I cells intersecting $E$ at its leftmost and rightmost subsegments, as in the previous case. Since $t = m$, we have $\mu = \tau$, hence $e_t = 0$.

The type-I mixed cells in any of the above mixed subdivisions vanish when $b_{tL} = 0$ or $b_{mR} = u$. Notice that $e_i + e_j + e_\lambda = u$ and since $e_\lambda$ is maximized, $(e_0, e_1, e_2)$ defines a vertex of $\mathcal{N}(\Phi) \subset \mathbb{R}^3$. □

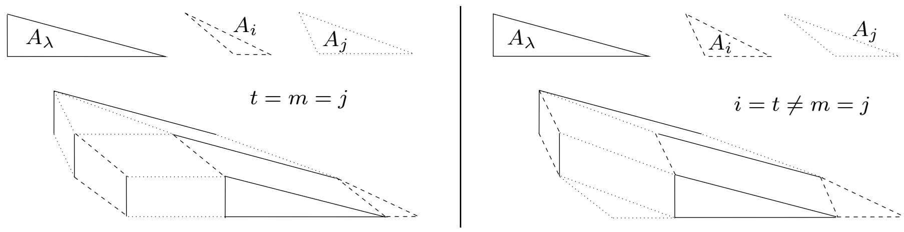


FIGURE 6. Lemma 18: the mixed subdivisions for a certain choice of $B_i$'s and cases $t = m$ and $t \neq m$.

The following corollary is proven similarly.

**Corollary 19.** *Under the notation of lemma* 18 *consider the following definition:*

1. $b_{tL} = \min\{b_{iL}, b_{jL}\},\ b_{mR} = \min\{b_{iR}, b_{jR}\}$*, provided* $b_{\lambda R} = u$.
2. $b_{tL} = \max\{b_{iL}, b_{jL}\},\ b_{mR} = \max\{b_{iR}, b_{jR}\}$*, provided* $b_{\lambda L} = 0$.
3. $b_{tL} = \max\{b_{iL}, b_{jL}\},\ b_{mR} = \min\{b_{iR}, b_{jR}\}$*, provided* $b_{tL} \leq b_{mR}$*,* $b_{\lambda R} = u$ *and* $b_{\lambda L} = 0$.

*Let* $\Delta = [\delta_{tL}, \delta_{mR}]$ *and in each case, define* $e_0, e_1, e_2$ *as in lemma* 18*. Then,* $(e_0, e_1, e_2)$ *is a vertex of* $\mathcal{N}(\Phi)$.

### 4.1. The implicit vertices

Overall, there are three cases for the relative positions of the $B_i$:

1. $\mathrm{CH}(B_i \cup B_j) = [0, u]$ for all pairs $i, j$.
2. $\mathrm{CH}(B_j \cup B_l) = \mathrm{CH}(B_i \cup B_l) = [0, u] \neq \mathrm{CH}(B_i \cup B_j)$.
3. $\mathrm{CH}(B_i \cup B_j) = [0, u] \neq \mathrm{CH}(B_l \cup B_t)$ for $t = i, j$.

Orthogonally, we can distinguish the following two cases:
(A) there exists at least one $\mathrm{CH}(B_i) = [0, u]$,
(B) none of the $B_i$'s satisfies $\mathrm{CH}(B_i) = [0, u]$.

In case (B), every union $B_i \cup B_j$ contains either 0 or $u$. Cases (1B) and (3A) cannot exist, which leaves 4 cases overall. In the sequel, we let $E_{it}$ denote a segment $(a_{i0}, a_{it}) \subset B_i$.

**Theorem 20.** [case (A)] *Recall that* $u = \max\{b_{0R}, b_{1R}, b_{2R}\}$*. If all unions CH*$(B_i \cup B_j) = [0, u], i \neq j$*, then the implicit polygon* $\mathcal{N}(\phi)$ *is a triangle with vertices* $(0, 0), (0, u), (u, 0)$*. Otherwise, if exactly one support, say* $B_k, k \in \{0, 1, 2\}$*, equals*

*$[0, u]$, then $\mathcal{N}(\phi)$ has up to 5 vertices $(e_0, e_1)$ which can be read of from the following set of $(e_i, e_j, e_k)$ vectors:*

$$\{((u,0,0),(0,u,0),(0,u-b_{iR}+b_{iL},b_{iR}-b_{iL}),(b_{jL},u-b_{iR},0),\\ (u-b_{jR}+b_{jL},0,b_{jR}-b_{jL})\},$$

*where $\{i,j,k\}=\{0,1,2\}$, assuming $i,j$ are chosen so that*

$$b_{iL}(u-b_{jR}) \geq b_{jL}(u-b_{iR}). \tag{10}$$

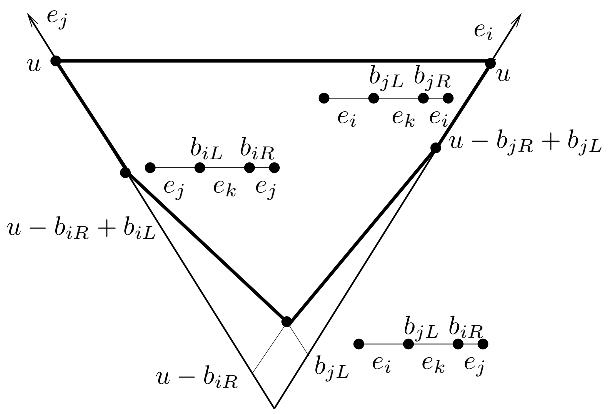


FIGURE 7. The implicit polygon in case (2A), in the $e_ie_j$-plane, and the subdivisions of the proof of theorem 20.

*Proof.* Case (1A) is established by lemma 18. Consider case (2A): By switching $i$ and $j$, assumption (10) can always be satisfied. Unless $B_i \subset B_j$ or $B_j \subset B_i$, this assumption holds simply by choosing $i, j$ so that $b_{jL} \leq b_{iL}$.

The vertices $(u,0,0),(0,u,0)$ are obtained by lemma 18, applied to $\mathrm{CH}(B_j \cup B_k)$ and $\mathrm{CH}(B_i \cup B_k)$ respectively. The third point is obtained by a mixed subdivision with two type-I cells $E_{iL}+a_{j0}+E_{kL}, E_{iR}+a_{j0}+E_{kR}$, which contribute the lengths of $[b_{kL}, b_{iL}], [b_{iR}, b_{kR}]$ to $e_j$, and one type-II cell $E_{i0}+E_{jt}+a_{k0}$, contributing the length of $[b_{iL}, b_{iR}]$ to $e_k$, where $E_{i0}$ is the horizontal edge of $A_i$ and $t \in \{L, R\}$; see figure 7. By switching $i$ and $j$ we define a subdivision that yields the fifth point.

The fourth point is obtained by a subdivision with 3 type-I cells: $a_{i0}+E_{jL}+E_{kL}, E_{iR}+a_{j0}+E_{kR}$ and $E_{iR}+E_{jL}+a_{k0}$, which contribute to $e_i, e_j$ and $e_k$ respectively, see fig. 7. It suffices to show that the line defined by this and the third point supports the implicit polygon. An analogous proof then shows that the line defined by this and the 5th point also supports the polygon, and the theorem follows. Our claim is equivalent to showing

$$\det \begin{bmatrix} b_{jR} & u-b_{iR} & 1 \\ 0 & u-b_{iR}+b_{iL} & 1 \\ e_i & e_j & 1 \end{bmatrix} \leq 0 \Leftrightarrow b_{iL}(e_i - b_{jL}) \geq b_{jL}(u-b_{iR}-e_j). \tag{11}$$

We consider the rightmost subsegment on $E$, where one endpoint is $b_{kR} = u$. This contributes to either $e_i$ or $e_j$ an amount equal to the length of a subsegment extending at least as far left as $b_{jR}$ or $b_{iR}$, respectively. Symmetrically, the leftmost subsegment has endpoint $b_{kL} = 0$ and contributes to $e_i$ or $e_j$ the length of a subsegment extending at least as far right as $b_{jL}$ or $b_{iL}$, respectively. In general, there are 4 cases, depending on the contribution of the rightmost and leftmost subsegments. The last case is infeasible if $B_i, B_j$ have no overlap.

If the rightmost subsegment contributes to $e_j$ then $e_j \geq u - b_{iR}$. If the leftmost subsegment contributes to $e_j$ then this contribution is at least $b_{iL}$, hence $e_j \geq u - b_{iR} + b_{iL}$, where $e_i \geq 0$. Otherwise, the leftmost subsegment contributes to $e_i$, thus $e_i \geq b_{jL}$. In both cases, inequality (11) follows.

If the rightmost subsegment contributes to $e_i$ then $e_i \geq u - b_{jR}$. If the leftmost subsegment also contributes to $e_i$, then $e_i \geq u - b_{jR} + b_{jL}$. Using also $e_j \geq 0$, it suffices to prove $b_{iL}(u - b_{jR}) \geq b_{jL}(u - b_{iR})$. Otherwise, the leftmost subsegment contributes to $e_j$, so $e_j \geq b_{iL}$, and it suffices to prove $b_{iL}(u - b_{jR} - b_{jL}) \geq b_{jL}(u - b_{iR} - b_{iL})$. Both sufficient conditions are equivalent to assumption (10). □

**Theorem 21.** [case (B)] *Recall that* $u = \max\{b_{0R}, b_{1R}, b_{2R}\}$. *If none of the* $B_t$*'s is equal to* $[0, u]$, *then we may choose* $\{i, j, k\} = \{0, 1, 2\}$ *such that:*

$$0 < b_{iL} \leq b_{iR} = u,\ 0 = b_{jL} \leq b_{jR} < u,\ 0 \leq b_{kL} \leq b_{kR} \leq u,\ B_k \neq [0, u].$$

*Then,* $\mathcal{N}(\phi)$ *has at most 5 or 4 vertices, depending on whether* $b_{kL}$ *is positive or 0. In the former case, the vertices* $(e_0, e_1)$ *can be read of from the following set of* $(e_i, e_j, e_k)$ *vectors:*

$$\{(b_{jR}, 0, u - b_{jR}), (b_{kR}, u - b_{kR}, 0), (b_{kL}, u - b_{kL}, 0), (0, u - b_{iL}, b_{iL}), (0, 0, u),\}$$

*and, in the latter case, the third and fourth vertices are replaced by* $(0, u, 0)$.

By lemma 13, at every point $e_k = u - e_i - e_j$. The theorem is established by the following two lemmas.

**Lemma 22.** [case (2B)] *Suppose* $b_{kL} = 0$ *in theorem* 21 *and w.l.o.g. assume* $b_{jR} \leq b_{kR}$. *Then,* $\mathcal{N}(\phi)$ *has up to 4 vertices* $(e_0, e_1)$ *which can be read of from the following set of* $(e_i, e_j, e_k)$ *vectors:*

$$\{(b_{jR}, 0, u - b_{jR}), (b_{kR}, u - b_{kR}, 0), (0, u, 0), (0, 0, u)\}.$$

*Proof.* The last two vertices follow from lemma 18, applied to $B_i, B_k$ and $B_i, B_j$, respectively. The same lemma, applied to $B_j, B_k$, yields the second vertex and cor. 19 applied to $B_j, B_k$, yields the first vertex. It suffices to show that any point $(e_i, e_j) \in N(\phi)$ defines a counter-clockwise turn in the $e_ie_j$-plane, when appended to $(b_{jR}, 0)$ and $(b_{kR}, u - b_{kR})$. This is equivalent to proving

$$\det \begin{bmatrix} b_{jR} & 0 & 1 \\ b_{kR} & u - b_{kR} & 1 \\ e_i & e_j & 1 \end{bmatrix} \geq 0 \Leftrightarrow e_j(b_{kR} - b_{jR}) \geq (u - b_{kR})(e_i - b_{jR}). \qquad (12)$$

Rightmost segment $[b_{kR}, b_{iR} = u]$ cannot contribute to $e_i$, since each corresponding mixed cell has an edge summand from $A_i$. If the segment lies in a $j$-mixed cell, then $e_j \geq u - b_{kR}$ and $e_i \leq b_{kR}$, and inequality (12) is proven. Otherwise, at least a subsegment contributes to a $k$-mixed cell.

If this subsegment contains $b_{kR}$, then it must extend at least to the next endpoint lying left of $b_{kR}$, hence to $b_{jR}$ or $b_{iL}$. In the latter case, the subsegment to the left of $b_{iL}$ cannot contribute to $e_i$. In any case $e_i \leq b_{jR}$, so (12) is proven.

If none of the above happens, then the subsegment contributing to $e_k$ does not contain $b_{kR}$, so the only way for the $k$-mixed cell to be defined is to have $b_{iL}$ lie in $(b_{kR}, b_{iR})$ and $k$-mixed cell intersecting $E$ at $[b_{iL}, b_{iR}]$. Then, $[b_{kR}, b_{iL}]$ contributes to $e_j$, so the $j$-mixed cell intersects $E$ at $[b_{kt}, b_{iL}]$, where $t \in \{L, R\}$. If $b_{kt} = b_{kL}$, then $e_i = 0$ and (12) is proven.

Otherwise, $b_{kt} = b_{kR}$. The $j$-mixed cell is of type I and implies that the 1-dimensional face $(b_{iL}, 2) + E_{kR}$ belongs to the subdivision, see lemma 12. The $k$-mixed cell is of type II, with some edge summand $E_{jt} \subset A_j$, which implies that the 1-face $(b_{iL}, 2) + E_{jt}$ is in the subdivision and cannot lie to the left of the previous 1-face. Since $b_{jR} \leq b_{kR}$, we have $b_{kR} = b_{jR}$, hence $e_i \leq b_{jR}$. □

**Lemma 23.** [case (3B)] *Suppose* $b_{kL} > 0$ *in theorem* 21*. Then,* $\mathcal{N}(\phi)$ *has up to 5 vertices* $(e_0, e_1)$ *which can be read of from the following set of* $(e_i, e_j, e_k)$ *vectors:*

$$\{(b_{jR}, 0, u - b_{jR}), (b_{kR}, u - b_{kR}, 0), (b_{kL}, u - b_{kL}, 0), (0, u - b_{iL}, b_{iL}), (0, 0, u)\}.$$

*Proof.* The last vertex follows from lemma 18, applied to $B_i, B_j$. We shall prove that the first two points are vertices. When $b_{jR} \geq b_{kR}$, the first point is obtained by lemma 18 applied to $B_j, B_k$, and the second one by cor. 19 applied to $B_j, B_k$, and vice versa when $b_{jR} < b_{kR}$. The third and fourth vertices are established analogously, by considering $B_i, B_k$.

Our proof shall establish inequality (12). If $b_{jR} \leq b_{kR}$, this is similar to the proof of lemma 22. Otherwise, $b_{kR} < b_{jR}$, and the rightmost segment $[b_{jR}, b_{iR} = u]$ cannot contribute to $e_i$. If it contributes to $e_k$ only, then $e_k \geq u - b_{jR}$ so $e_i + e_j \leq b_{jR}$ and (12) follows.

If it contributes to $e_j$ only, the union of the corresponding $j$-mixed cells intersect $E$ at a segment with an endpoint to the left of $b_{jR}$, namely $b_{kt},\ t \in \{L, R\}$, or $b_{iL}$. In the former case, $e_i \leq b_{kR}$ and $e_j \geq u - b_{kR}$. In the latter case, $[0, b_{iL}]$ contributes to $e_k$ only, so $e_i = 0, e_j = u - b_{iL}$. In both cases, (12) follows readily.

Lastly, $[b_{jR}, b_{iR}]$ might be split into subsegments $[b_{jR}, b_{iL}], [b_{iL}, b_{iR}]$, contributing to $e_k, e_j$ respectively. The corresponding cells are of type I and type II, the latter having an edge summand from $A_k$. This requires the subdivision to have $j$-faces $(b_{iL}, k) + E_{jR}$ and $(b_{iL}, 2) + E_{kt}, t \in \{L, R\}$, where the first lies to the left of the second, see lemma 12. This cannot happen because $b_{kR} < b_{jR}$. □

Now we consider the case of polynomial parameterizations $x = P_0(t),\ y = P_1(t)$. Let $B_i = \{b_{iR}, \ldots, b_{iR}\},\ i = 0, 1$, be the supports of polynomials $P_0, P_1$. The following is an immediate corollary of theorems 20 and 21 when $B_2 = \{0\}$.

**Corollary 24.** *If $P_0$ or $P_1$ (or both) contain a constant term, then the implicit polygon is the triangle with vertices $(0,0),(b_{1R},0),(0,b_{0R})$. Otherwise, $P_0,P_1$ contain no constant terms, and the implicit polygon is the quadrilateral with vertices $(b_{1L},0),(b_{1R},0)$, $(0,b_{0R}),(0,b_{0L})$.*

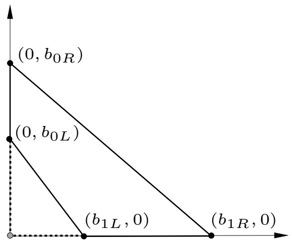


FIGURE 8. The implicit polygon of a polynomially parameterized curve.

We use [GKZ90, prop. 15] to arrive at the following; recall that the implicit equation is defined up to a sign. Let $c \in \{-1,1\}$; the coefficient of $x^{a_{1m}}$ is $c(-1)^{(1+a_{0n})a_{1m}}c_{1m}^{a_{0n}}$ and that of $y^{a_{0n}}$ is $c(-1)^{a_{0n}(1+a_{1m})}c_{0n}^{a_{1m}}$.

**Corollary 25.** *There exists $c \in \{-1,1\}$ s.t. the coefficient of $x^{a_{1m}}$ is $c(-c_{1m})^{a_{0n}}$ and that of $y^{a_{0n}}$ is $c(-c_{0n})^{a_{1m}}$.*

We give certain examples of polynomial parameterizations, all leading to optimal implicit supports.

**Example 7.** Parameterization $x = y = t$ yields implicit equation $\phi = x - y$. Our method yields vertices $(1,0)$ and $(0,1)$ which are optimal.

Parameterization $x = 2t^3 - t + 1, y = t^4 - 2t^2 + 3$ yields implicit equation $\phi = 608 - 136x + 569y + 168y^2 - 72x^2 - 32xy - 4x^3 - 16x^2y - x^4 + 16y^3$. Our method yields the vertices $(0,0),(4,0),(0,3)$ which are optimal. The degree bounds describe a larger quadrilateral with vertices $(0,0),(4,0),(1,3),(0,3)$. Corollary 25 predicts, for $x^4$, coefficient $(-1)^{16} = 1$, and for $y^3$, coefficient $(-1)^{15}2^4 = -16$, up to a fixed sign which equals $-1$ in $\phi(x,y)$.

For the Fröberg-Dickenstein example [EK05, Exam.3.3],

$$x = t^{48} - t^{56} - t^{60} - t^{62} - t^{63}, y = t^{32},$$

our method yields vertices $(32,0),(0,48),(0,63)$, which define the optimal polygon. Here the degree bounds describe the larger quadrilateral with vertices $(0,0),(32,0)$, $(32,31),(0,63)$. Parameterization $x = t + t^2, y = 2t - t^2$ yields implicit equation $\phi = 6x - 3y + x^2 + 2xy + y^2$. Corollary 24 yields vertices $(1,0),(2,0),(0,2),(0,1)$, which define the actual implicit polygon. Here the degree bounds imply a larger triangle, with vertices $(0,0),(2,0),(0,2)$. Corollary 25 predicts, for $x^2$ and $y^2$, coefficients $(-1)^6(-1)^2 = 1$ and $(-1)^6(1)^2 = 1$ respectively.

**Example 8.** [Cont'd from ex. 4] For the unit circle, $x = 2t/(t^2+1),\ y = (1-t^2)/(t^2+1)$, we have $f_0 = xt^2 - 2t + x,\ f_1 = (y+1)t^2 + (y-1)$. In lemma 18, the sets $B_0 = \{1\}, B_1 = \{0,2\}, B_2 = \{0,2\}$ yield implicit vertices $(2,0),(0,2),(0,0)$, corresponding to terms $x^2, y^2, 1$ in $\phi$ and, hence, an optimal support. See example 4 for a treatment assuming different denominators.

**Example 9.** [Cont'd from ex. 6] For the folium of Descartes

$$x = \frac{3t^2}{t^3+1}, y = \frac{3t}{t^3+1} \Rightarrow \phi = x^3 + y^3 - 3xy = 0;$$

see figure 4. Now, $B_0 = \{2\}, B_1 = \{1\}, B_2 = \{0,3\}$, hence this is case (A). In theorem 20, we set $i = 0, j = 1, k = 2$ and obtain, implicit vertices in the order stated by the theorem: $(3,0),(0,3),(1,1),(0,3)$ corresponding to terms $x^3, y^3, y^3, xy, x^3$, hence an optimal support.

If we do not account for the same denominators, use degree bounds alone, or project the Sylvester resultant, we obtain the additional vertex $(0,0)$ which leads to a support with 5 extra points.

**Example 10.**

$$x = \frac{2t^3+t+1}{t^2+1}, y = \frac{t^4+t^3-1}{t^2+1},$$

hence $B_0 = \{0,1,3\}, B_1 = \{0,3,4\}, B_2 = \{0,2\}$, so this is case (2A) with $B_1 = [0,u]$. In theorem 20, we set $i = 0, j = 2$ and obtain the vectors $(e_i, e_j) = (e_0, e_2) = (4,0),(0,4),(0,1),(0,3),(2,0)$, in the order stated by the theorem. This yields the implicit points $(e_0, e_1) = (4,0),(0,0),(0,3),(0,1),(2,2)$, hence vertices $(4,0),(0,0),(0,3),(2,2)$. These define the optimal polygon because the implicit equation is $\phi = 59 - 21x + 110y + 52y^2 - 13x^2 - 48xy + 5x^3 - 5x^2y - x^4 + 8y^3 - 2x^2y^2 + 2x^3y - 12xy^2$. If we do not exploit the identical denominators and use the method for different denominators, we obtain points $(4,2),(2,3),(4,0),(0,0)$ and $(0,3)$ which define a polygon that contains the implicit polygon. Taking into account the degree bound (total degree=4), rules out points $(4,2)$ and $(2,3)$, and introduces point $(1,3)$, yielding a smaller polygon that still contains the implicit polygon.

**Example 11.** [Cont'd from ex. 1]

$$x = \frac{t^6+2t^2}{t^7+1}, y = \frac{t^4-t^3}{t^7+1},$$

hence $B_0 = \{2,6\}, B_1 = \{3,4\}, B_2 = \{0,7\}$, so this is case (2A) with $B_2 = [0,u]$. In theorem 20, we set $i = 0, j = 1$ and obtain the implicit points $(e_0, e_1) = (7,0),(0,7),(0,3),(3,1),(6,0)$, in the order stated by the theorem. The first 3 points follow from lemma 18, while the last 2 follow from corollary 19(2) and (3) respectively, applied to $B0, B1$. These are also the implicit vertices and define the actual polygon because the implicit equation is eq. (1). In figure 1 is shown the implicit polygon. Changing the coefficient of $t^2$ to $-1$, leads to an implicit polygon with 6 vertices $(1,3),(0,4),(0,6),(2,5),(7,0),(4,1)$, is contained in the polygon predicted by theorem 20. This shows the importance of the genericity condition on the coefficients of the parametric polynomials.

## 5. Conclusion and Further work

In conclusion, we have proven that in all cases only the extremal terms matter, both in determining the implicit polygon as well as in ensuring the genericity hypothesis on the coefficients.

It is possible to use our results in deciding which polygons can appear as Newton polygons of plane curves, and which parameterization is possible in the generic case. In particular, corollary 24 and corollary 25 imply that the Newton polygon of polynomial curves always has one vertex on each axis. These vertices define the edge that equals the polygon's upper hull in direction $(1, 1)$. The rest of the edges form the lower hull. If the implicit polygon is a segment, then the implicit polygon cannot contain interior points. Similar results hold for curves parameterized by Laurent polynomials.

We have shown that the case of common denominators reduces to a particular system of 3 bivariate polynomials, where only *linear* liftings matter. An interesting open question is to examine to which systems of general dimension this observation holds, since it simplifies the enumeration of mixed subdivisions and, hence, of the extreme resultant monomials. In particular, we may ask whether this holds whenever the Newton polytopes are pyramids, or for systems with separated variables.

Another interesting question is whether we can extend our methods to the implicit polytope of a rational surface. Lastly, by approximating the given polygon by one of the polygons described above, one might formulate a question of approximate parameterization.

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