Computing the Newton Polytope of the Resultant

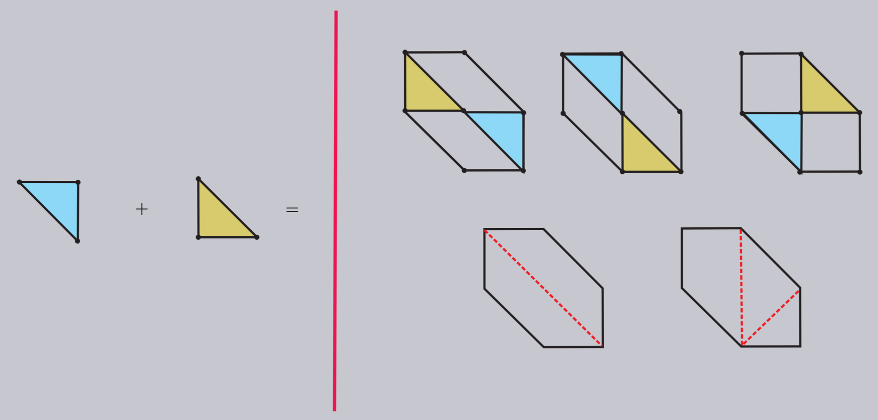
Ioannis Z. Emiris, Christos Konaxis
Department of Informatics and Telecommunications,
University of Athens

Mixed Subdivisions

- The ***support*** $A(f)$ of a polynomial f is the set of the exponent vectors of its monomials with nonzero coefficients. The ***Newton polytope*** $N(f)$ of f is the convex hull of its support.
- Let $f_0, \dots, f_n$, be $n+1$ Laurent polynomials in $\mathbb{C}[x_1, \dots, x_n]$ with symbolic coefficients $c_{i,j}$ and Newton polytopes $P_0, \dots, P_n \subset \mathbb{R}^n$. Suppose $P = P_0 + \dots + P_n \subset \mathbb{R}^n$, is a n-dimensional convex polytope.
- A ***tight mixed subdivision*** of P, is a collection of n-dimensional convex polytopes R, called ***cells***, st.:
  1. They form a polyhedral complex that partitions P and
  2. Every cell R is a Minkowski sum of faces of the polytopes P_i:

$$R = F_0 + \dots + F_n, \quad dim(R) = dim(F_0) + \dots + dim(F_n) = n,$$

- **Definition.** A cell R is called i-***mixed*** if it is a Minkowski sum of n edges $E_j \subset P_j$ and one vertex $v_i \in P_i$:
$$R = E_0 + \dots + v_i + \dots + E_n.$$
- A mixed subdivision is called regular if it can be obtained from the projection of the lower hull of the Minkowski sum of lifted polytopes $\hat{P}_i := \{(p_i, \omega_i(p_i)) \mid p_i \in P_i\}$. If ω_i is generic, the induced mixed subdivision is tight.
- Two mixed subdivisions are equivalent if they have the same mixed cells. We call the equivalence classes ***mixed cell configurations***.



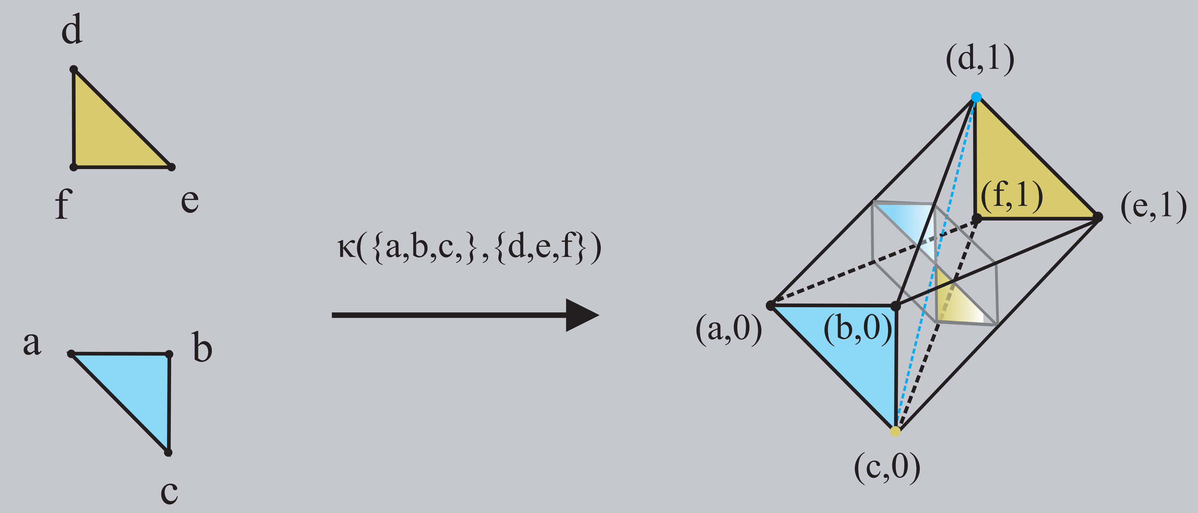
Mixed and not mixed subdivisions of the Minkowski sum of two triangles.

The Newton Polytope of the Sparse Resultant

- **Definition.** The toric or sparse resultant $\mathcal{R}$ of polynomials f_i, $i = 0, \dots, n$, is the unique (up to sign) irreducible polynomial in $\mathbb{Z}[c_{i,j}]$ which vanishes iff the f_i have a common root in $(\mathbb{C}^*)^n$.
- A monomial of the sparse resultant is called ***extreme*** if its exponent vector is a vertex of the Newton polytope $N(\mathcal{R})$ of the resultant.
- **Theorem.** (Sturmfels) For every generic lifting function ω, we obtain an extreme monomial of $\mathcal{R}$, of the form
$$init_\omega(\mathcal{R}) = c \cdot \prod_{i=0}^{n} \prod_{R} c_{i,v_i}^{\mathrm{Vol}(R)},$$
where the second product is over all i-mixed cells R of the regular tight mixed subdivision of $P = \sum_{i=0}^{n} P_i$, induced by ω and c_{i,v_i} is the coefficient of the monomial of f_i corresponding to the vertex v_i. The constant c is +1 or -1.
- **Corollary.** There exists a 1-1 and onto correspondence between the extreme monomials and the mixed cell configurations.

The Cayley Trick

- Given supports $A_0, \dots, A_n$, the Cayley embedding κ introduces a new point set
$$C := \kappa(A_0, A_1, \dots, A_n) = \bigcup_{i=0}^{n} (A_i \times \{e_i\}) \subset \mathbb{R}^{2n+1},$$
where e_i are an affine basis of $\mathbb{R}^n$. The dimension of the convex hull of C is $d := 2n$.

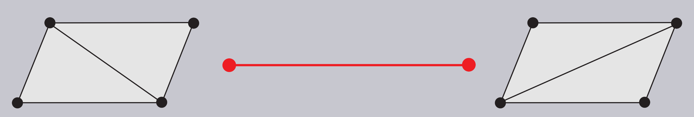


The image via κ of two triangles.

- **Theorem.** (The Cayley Trick) There exists a bijection between the tight regular mixed subdivisions of the Minkowski sum P and the regular triangulations of C.

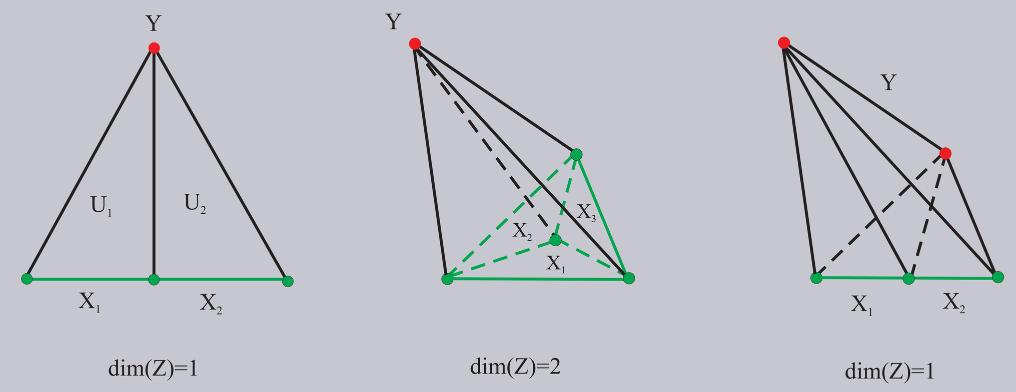
Enumeration of Mixed Cell Configurations

- Regular triangulations of C are in bijection to the vertices of the so called ***secondary polytope*** $\Sigma(C)$ of C. Two vertices in $\Sigma(C)$ are connected by an edge if they can be obtained from each other by a local modification called ***bistellar flip***.



Secondary polytope of a quadrilateral.

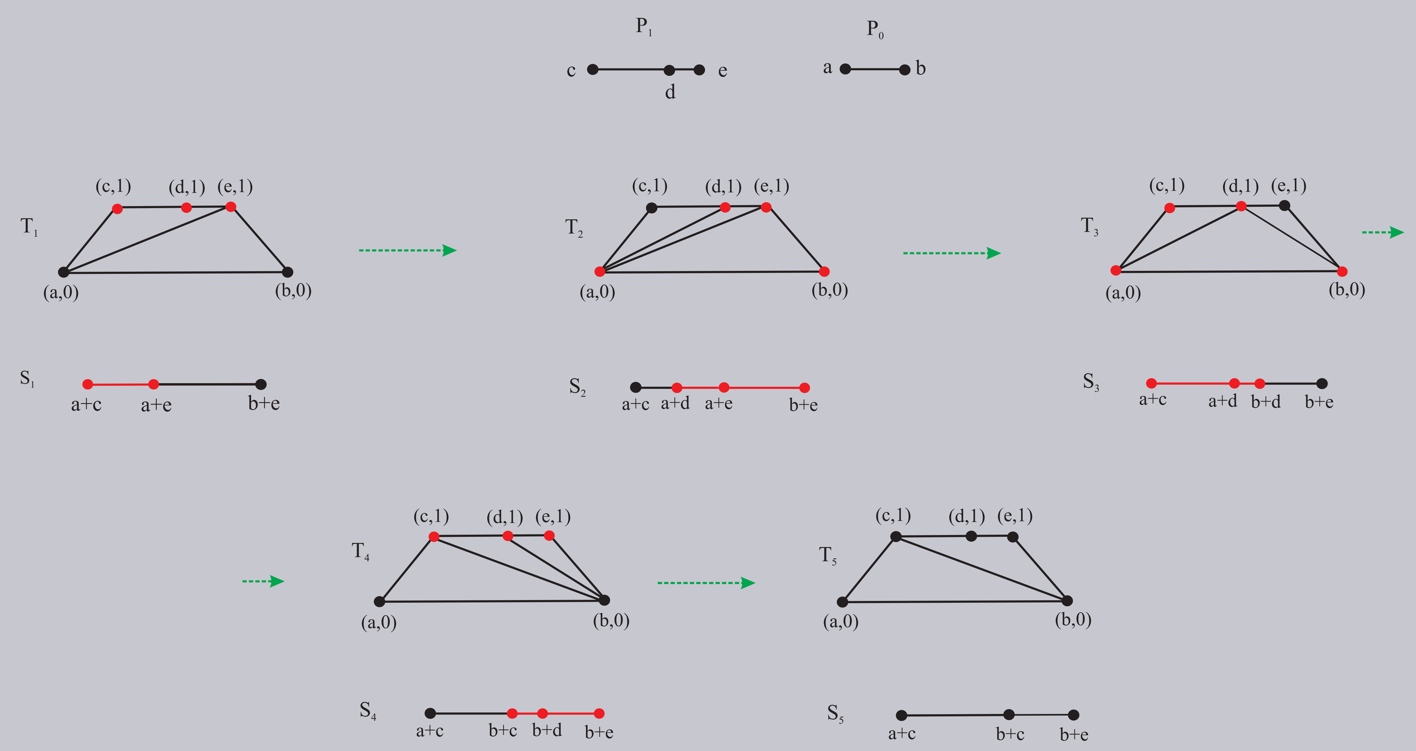
- One can enumerate all regular triangulations of C by computing a spanning tree of the secondary polytope $\Sigma(C)$. The algorithm proposed by Imai et. al.[2003] uses reverse search for low memory usage.
- We allow bistellar flips only on suitable circuits, thus obtaining a regular triangulation corresponding to a new mixed cell configuration.
- The suitable circuits are characterized by cardinality (odd and even circuits).



Odd circuits (left and right figures) and a non suitable circuit.

An Example

- $f_0 = c_{0,1}x^a + c_{0,2}x^b$, $f_1 = c_{1,1}x^c + c_{1,2}x^d + c_{1,3}x^e \in \mathbb{C}[x]$.
- The supports A_0, A_1, the point set $C = \kappa(A_0, A_1)$ and the enumeration of the regular triangulations of C corresponding to the mixed cell configurations of $P = P_0 + P_1$, are shown below. The circuits on which we perform bistellar flips are depicted in red.



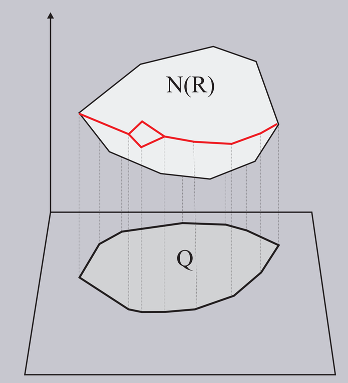
An Application to Implicitization

Input: Parametric representation of a hypersurface $x_i = \frac{P_i(t)}{Q(t)}$, $i = 0, \dots, n$, $\gcd(P_i(t), Q(t)) = 1$.
Output: A superset of the support of the implicit equation.

1. Define $f_i = x_i Q(t) - P_i(t)$ as polynomials in t: $f_i = \sum c_{ij} t^{a_{ij}} \in \mathbb{C}[t]$, $c_{i,j}$ generic coefficients.
2. Compute the extreme monomials of the resultant of f_i using our algorithm. Then compute a superset of the support of the resultant.
3. Transform the set of monomials of the form $\prod c_{ij}^{e_{ij}}$, to a set of monomials in the x_i. This is equivalent to projecting the Newton polytope of the resultant of f_i onto a 2 or 3-dimensional subspace ($n = 2$ or 3).

Future Work

- Enumerate only the vertices of the secondary polytope $\Sigma(C)$ that correspond to mixed cell configurations lying on the ***silhouette*** of $N(\mathcal{R})$ with respect to a canonical projection π.
- Equivalently: characterize the circuits that lead to a new vertex on the silhouette.



The projection of step 3 of the implicitization algorithm.