Approximate Price of Anarchy and Stability

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Barcelona 2009/11/19

[1] Joint work with George Christodoulou and Paul Spirakis

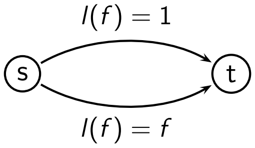
The price of anarchy

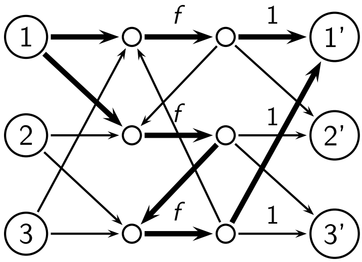
Price of anarchy

- Equilibria (Nash, correlated, etc) optimize the utility of each player individually.
- They don't optimize globally
- Price of Anarchy $\rightarrow$ How suboptimal are the equilibria?

- Price of anarchy (PoA) - worst equilibrium
- Price of stability (PoS) - best equilibrium

Examples - Price of anarchy

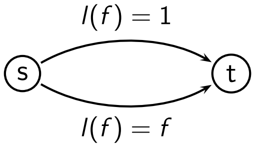


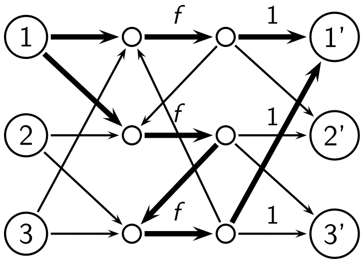


Price of anarchy

- The PoA of the Pigou example (on the left) is 4/3
- The PoA of the congestion game (on the right) is 2.

Examples - Price of stability





Price of stability

- The PoS of the Pigou example (on the left) is $4/3$
- The PoS of the congestion game (on the right) is 1.

Atomic congestion games

Definition (Atomic congestion game)

- n players
- m facilities (or edges)
- Each facility e has a cost or latency function ℓ_e: When k players use it, the cost is $\ell_e(k)$.
- Strategy is a set of facilities (or a path)
- For a strategy profile the cost of a player is the sum of the cost of the facilities in his strategy.

Atomic congestion games

Cost of a player

- Let $A = (A_1, \ldots, A_n)$ be strategies of the n players.
- $n_e(A) = |\{i : e \in A_i\}|$ denotes the number of players who use facility e
- The cost of player i is

$$c_i(A) = \sum_{e \in A_i} \ell_e(n_e(A))$$

- The PoA is

$$max_{A:\text{ equilibrium}} \frac{\sum_i c_i(A)}{\min_P \sum_i c_i(P)}$$

- The PoS is

$$min_{A:\text{ equilibrium}} \frac{\sum_i c_i(A)}{\min_P \sum_i c_i(P)}$$

Definition (Non-atomic congestion game)

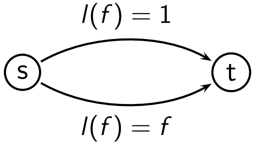
- It is the limit case of atomic games, when the number of players tends to infinity
- It is usually defined on a network
- Fixed rates of flow r_{ij} between pairs of nodes

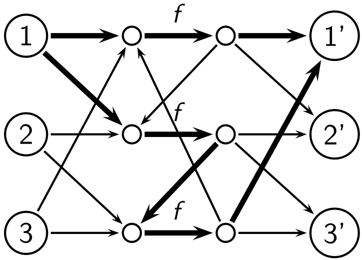
PoA and PoS

PoA and PoS of a class of games: What is the maximum PoA and PoS for the games in the class?

- The PoA and PoS are equal for non-atomic congestion games: There is a unique Nash equilibrium
- They may differ in atomic games (many equilibria)

Congestion games - Differences





- Atomic Congestion games: The number of players is finite
- Non-atomic congestion games: The number of players is infinite
- Major difference: In atomic games, when a player switches strategy it changes the cost of the facilities.

References

- Non-atomic congestion games have been studied for decades
- The atomic congestion games were introduced by Rosenthal in 1973
- The PoA of was introduced in 1999 (K-Papadimitriou), for simple weighted atomic games
- The PoA of non-atomic congestion games was first studied by Roughgarden and Tardos in 2000
- The PoS was first studied by Anshelevich et al in 2003 for atomic games with decreasing latency functions.
- The PoA and PoS of atomic games for lineal latencies was resolved in 2005 (Christodoulou-K, Awerbuch-Azar-Epstein)

Approximate Nash equilibria

- A set of strategies is an ϵ-Nash equilibrium when no player can gain more than ϵ by switching to another strategy
- Additive: $c_i(A) \leq c_i(A'_i, A_{-i}) + \epsilon$
- Multiplicative $c_i(A) \leq (1 + \epsilon) c_i(A'_i, A_{-i})$

Approximate Nash equilibria

- Exact Nash equilibria for arbitrary games is PPAD-complete (Daskalakis-Goldberg-Papadimitriou, Chen-Deng)
- Approximate Nash equilibria is in P for $\epsilon = 0.33$ and PPAD-complete for $\epsilon = 1/n$. Major open open.

Main questions

- The PoA (as a function of ϵ) increases. How?
- The PoS decreases. How?

By answering these questions we also

- recapture almost all known results about the PoA and PoS of atomic and non-atomic congestion games
- shed a new light on them

The price of anarchy

Theorem (No-atomic PoA, polynomial latencies)

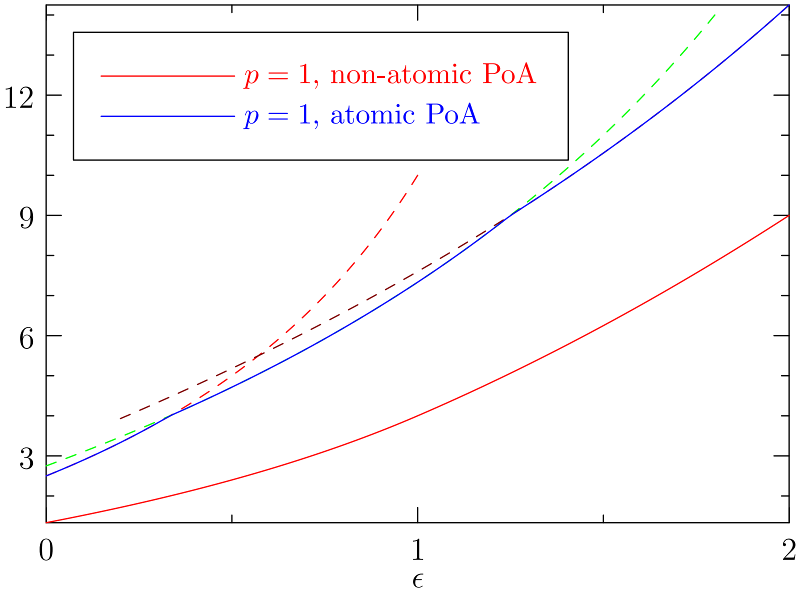
$$\begin{cases} (1+\epsilon)^{p+1} & \textit{for } \epsilon \geq (p+1)^{1/p} - 1 \\ (1/(1+\epsilon) - p/(p+1)^{1+1/p})^{-1} & \textit{otherwise} \end{cases}$$

Theorem (Atomic PoA)

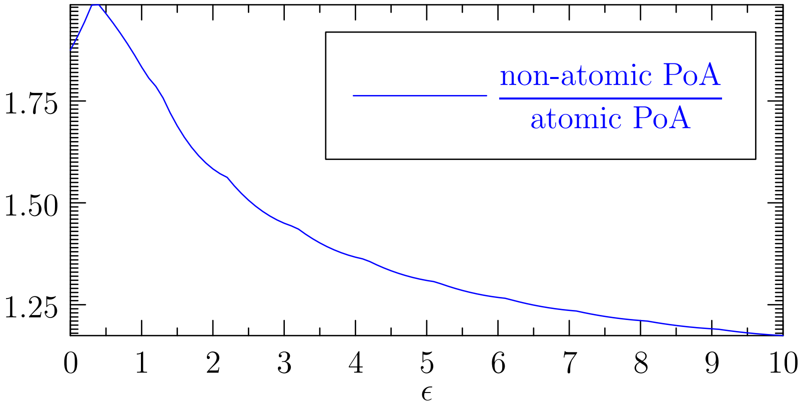
$$\frac{(1+\epsilon)\left((z+1)^{2p+1} - z^{p+1}(z+2)^p\right)}{(z+1)^{p+1} - (1+\epsilon)(z+2)^p + (1+\epsilon)(z+1)^p - z^{p+1}}$$

z is the maximum **integer** *with* $\frac{z^{p+1}}{(z+1)^p} \leq 1+\epsilon$.

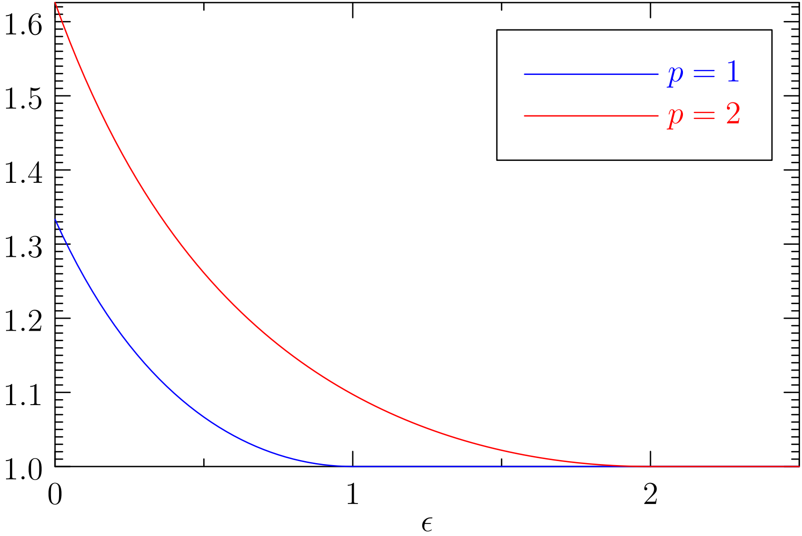
The price of anarchy



Non-atomic PoA vs Atomic PoA



PoS



$$\left((1+\epsilon)\left(1-\frac{p}{p+1}\left(\frac{1+\epsilon}{p+1}\right)^{1/p}\right)\right)^{-1}$$

PoS - Linear atomic games

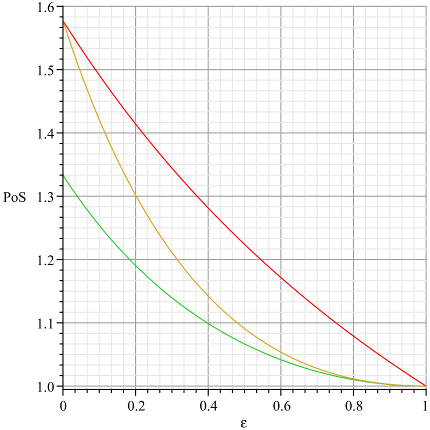


FIGURE: The upper and lower bound of the PoS. The lower line is the Pigou bound $\frac{4}{(1+\epsilon)(3-\epsilon)}$.

Proof for non-atomic PoA

- How to relate an ϵ-Nash flow f to some other feasible flow f^*?
- f^* is not necessarily optimal

Theorem

$$\left.\begin{array}{ll} f & \epsilon - Nash \\ f^* & feasible \end{array}\right\} \quad \Rightarrow \quad \sum_{e \in E} \ell_e(f_e) f_e \leq (1+\epsilon) \cdot \sum_{e \in E} \ell_e(f_e) f_e^*$$

Proof.

$$\epsilon - \text{Nash} \Rightarrow \sum_{e \in p} \ell_e(f_e) \leq (1+\epsilon) \cdot \sum_{e \in p'} \ell_e(f_e)$$

Sum this over all paths p and p', weighted by $f_p \cdot f_{p'}$. □

Non-atomic PoA (cont.)

To bound the PoA, the only game-theoretic fact we use is the inequality

$$\sum_{e \in E} \ell_e(f_e) f_e \leq (1+\epsilon) \cdot \sum_{e \in E} \ell_e(f_e) f_e^*$$

of the previous theorem!

The rest is based on two ideas:

- ignore the outer sum (the topology of the network)
- use an appropriate arithmetic inequality

Non-atomic PoA (cont.)

$$\ell_e(f_e)f_e \leq (1+\epsilon)\ell_e(f_e)f_e^*$$

We want to bound $\ell_e(f_e)f_e^*$ by a linear combination of $\ell_e(f_e)f_e$ and $\ell_e(f_e^*)f_e^*$ (the cost of the flows f and f^*).

We ask: For which α and β:

$$\ell_e(f_e)f_e^* \leq \alpha \cdot \ell_e(f_e)f_e + \beta \cdot \ell_e(f_e^*)f_e^*$$

Among all pairs that satisfy the inequality, minimize

$$\frac{(1+\epsilon)\beta}{1-(1+\epsilon)\alpha}$$

Non-atomic PoA (cont.)

For polynomials of degree p, $l_e(f_e) = f_e^p$:
We want to find α and β that satisfy

$$f_e^p f_e^* \leq \alpha f_e^{p+1} + \beta f_e^{*\,p+1}$$

for all nonnegative real values f_e and f_e^*, and minimize

$$\frac{(1+\epsilon)\beta}{1-(1+\epsilon)\alpha}$$

The solution to this program (with parameter ϵ) has different solutions for small ϵ and large ϵ.

$$\epsilon \geq (1+p)^{1/p} - 1 \quad \alpha = \frac{p}{(p+1)(1+\epsilon)} \quad \beta = (1+\epsilon)^p \quad \text{PoA} = (1+\epsilon)^{p+1}$$

$$\epsilon \leq (1+p)^{1/p} - 1 \quad \alpha = \frac{p}{(p+1)^{1+1/p}} \quad \beta = 1 \quad \text{PoA} = \ldots$$

Non-atomic PoA (cont.)

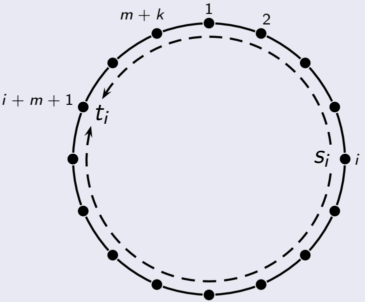
For $\epsilon = 0$, we recover the known results about the exact Nash equilibria. For example, for $\epsilon = 0$ and $p = 1$, we get

$$\alpha = \frac{1}{4} \quad \beta = 1 \quad \text{PoA} = \frac{4}{3},$$

the influential result of Roughgarden and Tardos.
This also shows that the PoA is (almost) independent of the network topology.

Non-atomic PoA - Lower bounds

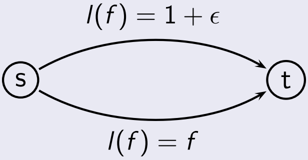
The fact that the price of anarchy is qualitatively different for small and large ϵ is reflected in the lower bounds too.
For **large** ϵ, the lower bound is given by the network



with $m/k \approx \epsilon$. The optimal routes counterclockwise and the equilibrium clockwise.

Non-atomic PoA - Lower bounds

For small ϵ, the situation is more revealing.
The lower bound for **exact equilibria** is given by the Pigou network:



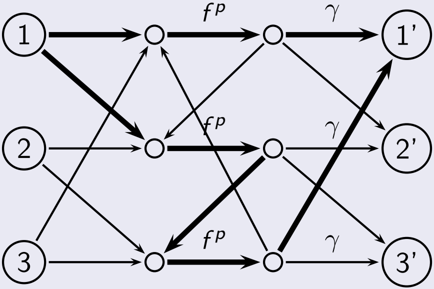
But this gives a lower bound of

$$\frac{4}{(1+\epsilon)(3-\epsilon)}$$

This is exact only for $\epsilon = 1$!

Non-atomic PoA - Lower bounds

In fact, a more complicated lower bound is needed for $\epsilon > 0$.



γ is a constant (which depends on ϵ).

How to bound PoS?

- Trying to apply the same approach directly to the PoS fails.
- We instead use a trick: Instead of bounding the PoS, we bound the PoA of **a subclass of strategies**. For this to work we need:
  - The subclass of strategies is guaranteed to contain an equilibrium
  - It is not very "large", so that we can get tight results
- Which subclass of strategies? **The ones that have minimum potential.**

Potential of atomic games

The potential of an atomic congestion game is defined by

$$\Phi_e(k) = \sum_{t=0}^{k} \ell_e(t)$$

$$\Phi(A) = \sum_{e \in E} \Phi_e(n_e(A))$$

Theorem

If f minimizes the potential then f is a Nash equilibrium.

Potential of non-atomic games

The potential of a non-atomic congestion game is defined a

$$\Phi_e(f_e) = \int_0^{f_e} \ell_e(t)\,dt$$
$$\Phi(f) = \sum_{e \in E} \Phi_e(f_e)$$

Theorem

If f minimizes the potential then f is a Nash equilibrium.

Using the potential to bound the PoS

- For exact equilibria, we can bound the PoS by bounding the PoA of the strategies that minimize the potential.
- This is the method implicitly or explicitly of almost all proofs about the PoS.
- But what is the potential for approximate equilibria?

Generalizing the potential - Non-atomic games

Let $\phi_e(f_e)$ be a function which satisfy

$$\frac{l_e(f_e)}{(1+\epsilon)} \leq \phi_e(f_e) \leq l_e(f_e),$$

For $\epsilon = 0$, $\phi_e = \ell_e$.
Define $\Phi_e(f_e) = \int_0^{f_e} \phi_e(t)\, dt$, and $\Phi(f) = \sum_{e \in E} \Phi_e(f_e)$.

Theorem

If a flow f minimizes the potential function $\Phi(f)$, it is an ϵ-Nash equilibrium.
Furthermore, when the latency functions are nondecreasing, for any other flow f':

$$\sum_{e \in E} \phi_e(f_e) f_e \leq \sum_{e \in E} \phi_e(f_e) f'_e$$

Bounding the PoS - Non-atomic games

The proof has the same structure with the proof about the PoA.

- Start with the inequality

$$\sum_{e \in E} \phi_e(f_e) f_e \leq \sum_{e \in E} \phi_e(f_e) f'_e$$

- Ignore the outer sum (the topology of the network)
- Decide what potential ϕ_e to use
- Determine the appropriate arithmetical lemma and apply it.

Bounding the PoS - Non-atomic games

When the latency functions are polynomials of degree p

$$\ell_e(f_e) = \sum_{k=0}^{p} a_{e,k} f_e^k$$

we need to decide what potential to use. We let

$$\phi_e(f_e) = \sum_{k=0}^{p} \zeta_k a_{e,k} f_e^k$$

for some ζ_k that satisfies $\frac{1}{1+\epsilon} \leq \zeta_k \leq 1$

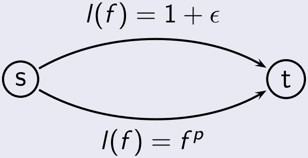
Arithmetical lemma

$$f_e^k f_e' \leq \alpha_k f_e^{k+1} + \beta_k f_e'^{k+1}$$

where $\alpha_k^k \beta_k = k^k/(k+1)^{k+1}$.

The PoS of non-atomic games

By selecting appropriate the parameters α_k, β_k, and ζ_k, we get the bound of the PoS.
It turns out that the answer is the PoS of the Pigou network.



Thus the Pigou network is the tight example for the PoS, not the PoA.

- Exactly the same techniques work for atomic games
- However, the proofs for the atomic games are technically more difficult.
- In fact, we need 2 types of inequalities for the PoS of atomic games:
  - A local one (as in the case of the PoA)
  - A global one: That the equilibrium has minimum potential (this is not necessary for the non-atomic case)
- The exact PoS stability is still open

Highlights

- Atomic games harder than non-atomic games
- PoS harder than PoA
- The results remain the same for mixed or correlated equilibria (we simply ignore the probabilities as we ignore the topology of the game).
- PoS drops to 1 when $\epsilon = p$.

Open problems about the PoS

- Determine the **exact** PoS for polynomial latencies when $p > 1$.
- Determine the **approximate** PoS for polynomial latencies (even for $p = 1$ is still open)
- Determine the PoS for decreasing latency functions.
- Determine the PoS in undirected graphs.

In particular, it is still open what is the PoS for undirected graphs with latencies $\ell_e(k) = 1/k$.
This was posed as an open problem in the first paper about the PoS (Anshelevich et al)!

Thank you