Contention resolution for congestion games

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Contention and Congestion

Congestion	Contention
When two or more users try to use the same resource, the cost is higher	When two or more users try to use a resource, nobody succeeds
Example: Congestion games / Internet routing	Example: Ethernet / wireless
Strategy: Set of resources.	Strategy of a user: Timing

- **In between:** The cost depends both on the set of selected resources and the timing.
- **Strategy:** Set of resources + Timing

Our game-theoretic abstraction

- The users play a congestion game but they also select the time to start.
- Each user decides which path to use and when. When users use the same link at the same time they incur a higher cost.

In this talk

- Congestion game: A set of parallel links with affine latencies.
- Affine latencies: When k users use link e, each one incurs cost $\ell_e(k) = a_e k + b_e$.
- Symmetric strategies

Two models

The boat model

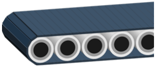
Only the group of players that start together affect the latency of the group.

- *At every time step*, a boat departs from the source of the link
- The speed of each boat depends only the number of players on it

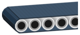
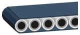
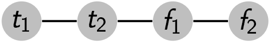
The conveyor belt model

The latency of a player depends on the number of other players using the link concurrently.

- The speed depends on the number of people on the belt



Details of the models

- Let $\ell_e(k)$ be the latency functions of the original congestion game
-  If a player decides to play at time t, he pays the original cost plus t
-  Each player has to complete a unit of work (or distance). Each time step, the player completes work $1/\ell_e(k)$ where k is the number of players using the same link.
-  Example with 2 players: 

$$\frac{t_2 - t_1}{\ell_e(1)} + \frac{f_1 - t_2}{\ell_e(2)} = 1$$
$$\frac{f_1 - t_2}{\ell_e(2)} + \frac{f_2 - f_1}{\ell_e(1)} = 1$$

Related work

Contention

- Game theoretic issues of Aloha / Slotted Aloha [MacKenzie-Wicker 2003, Altman-El Azouzi-Jimenez, 2004]. Time-invariant strategies.
- Time-dependent strategies for contention [Fiat-Mansour-Nadav 2007]. They study protocols with deadlines. They give a protocol which has low price of stability, with high probability.
- Extension to models with re-transmission cost [Christodoulou-Ligett-Pyrga 2010]

Related work

Congestion

- Atomic (finite number of players), non-atomic (infinite number of players / flow)
- Non-atomic congestion games have been studied for decades
- The atomic congestion games were introduced by Rosenthal in 1973
- The Price of Anarchy (PoA) of was introduced in 1999 (K-Papadimitriou), for simple weighted atomic games
- The PoA of non-atomic congestion games was first studied by Roughgarden and Tardos in 2000
- The Price of Stability (PoS) was first studied by Anshelevich et al in 2003 for atomic games with decreasing latency functions.
- The PoA and PoS of atomic games for linear latencies was resolved in 2005 (Christodoulou-K, Awerbuch-Azar-Epstein)

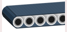
Related work

Game-theoretic analysis of TCP

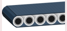
- [Akella-Seshan-Karp-Shenker-Papadimitriou 2002] studies TCP-like games. The strategies of a player are the parameters of AIMD which are not time-dependent.
- [Kesselman-Leonardi-Bonifaci 2005]. Game-theoretic issues of packet switching. It studies the steady state (strategies are the transmission rates).

Questions (and answers)

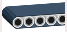
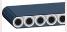
Are these congestion games?

-  Yes
-  Only for 2 players

Why?

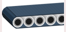
-  Take copies of the original game G_0, G_1, ... Add t to the latency functions of G_t.
-  For 2 players - 1 link: Take again copies of the original game G_0, G_1, ... Change the latencies of G_t as follows:

$$\ell'_{e_t}(1) = 1 + \left\lfloor \frac{t}{\ell_e(1)} \right\rfloor, \qquad \ell'_{e_t}(2) = \ell_{e_t}(1) + \frac{\ell_e(2) - \ell_e(1)}{\ell_e(1)}$$

  The player has to play $\ell_e(1)$ *consecutive* games.
-  For 2 players and arbitrary network, there is a potential function. Crucial: both players pay the same additive cost when they share a link.
-  for 3 or more players: There are games that have no pure (asymmetric) equilibria.

Questions (and answers)

Do they have pure equilibria?

-  Yes. Because they are congestion games.
-  Not in general. Even for the simple case of 1 link, 3 players, affine latencies ($\ell_e(k) = 5k - 1$).

 $\ell_e(k) = 5k - 1$

- Start times: $0 = t_1 \leq t_2 \leq t_3$. Finish times: f_1, f_2, f_3.
- Assume that they all overlap, i.e., $t_3 < f_1$. The other case is similar.

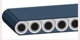
t_1 — t_2 — t_3 — f_1 — f_2 — f_3

$$\frac{t_2 - t_1}{\ell_e(1)} + \frac{t_3 - t_2}{\ell_e(2)} + \frac{f_1 - t_3}{\ell_e(3)} = 1$$
$$\frac{t_3 - t_2}{\ell_e(2)} + \frac{f_1 - t_3}{\ell_e(3)} + \frac{f_2 - t_3}{\ell_e(2)} = 1$$
$$\frac{f_1 - t_3}{\ell_e(3)} + \frac{f_2 - f_1}{\ell_e(2)} + \frac{f_3 - f_2}{\ell_e(1)} = 1$$

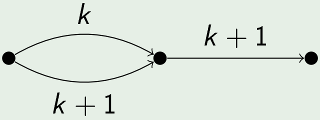
- We compute $f_3 = 14 - \frac{5}{36} t_2 - \frac{1}{9} t_3$. Best strategy for player 3: select $t_3 \geq f_1$ (no overlap).

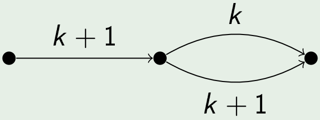
Questions (and answers)

Does the exact topology of the network matter?

-  No (as in congestion games)
-  Yes

Example

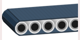
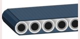




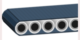
- Two players.
- On the left they finish at times $f_1 = 7/2$, $f_2 = 9/2$.
- One the right they finish at times $f_1 = 4$, $f_2 = 5$.

Questions (and answers)

What is the nature of the symmetric NE?

-   Unique symmetric NE
-  The probabilities NE drop linearly on every link
-  The probabilities are non-zero only at integral multiples of $\ell_e(1)$. At these times, they drop linearly.

What is the nature of the optimal symmetric solution?

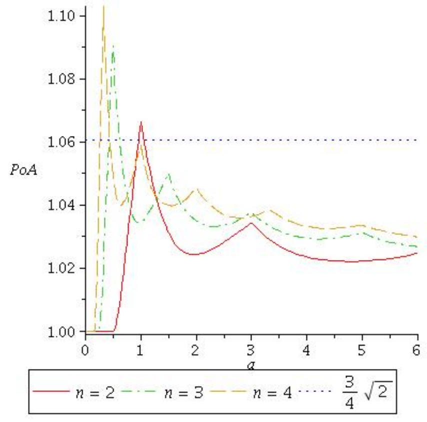
-   The optimal probabilities are identical to the Nash equilibrium of $\ell_e(k) = a_e^* k + b_e^*$ where

$$a_e^* = 2a_e \qquad b_e^* = b_e - a_e$$

Questions (and answers)

What is the PoA?

- It is small
- For fixed network, it tends to $3\sqrt{2}/4 \approx 1.06$ as the number of players tends to infinity.
-  For small number of players n and one link



The structure of the NE — Boat model

- The cost of a player who uses edge e at time t is

$$d_{e,t} = t + \sum_{k=0}^{n-1} \binom{n-1}{k} p_{e,t}^{k} (1 - p_{e,t})^{n-1-k} \ell_e(k+1)$$
$$= t + a_e + b_e + (n-1)\, a_e\, p_{e,t}$$

- NE if and only if: $p_{e,t} > 0$ implies $d_{e,t} = d = \min_{e,t} d_{e,t}$
- At a symmetric NE: $p_{e,t} \geq p_{e,t+1}$
- The support $\{t : d_{e,t} = d\}$ is $\{0, 1, \ldots, h_e\}$ for some **integer** h_e.
- The NE is the solution of the system

$d_{e,t} = d$ for $t \leq h_e$	They show that the probabilities drop linearly
$d_{e,h_e+1} > d$	It determines the parameters h_e as a function of the cost d
$\sum_{e,t} p_{e,t} = 1$	It determines d which happens to be unique

Optimal cost of symmetric strategies — Boat model

- The optimal cost L_{OPT} is the minimum of

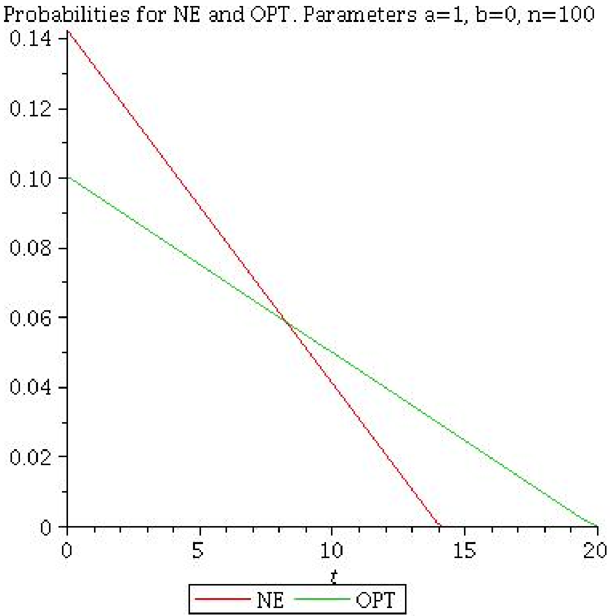
$$\sum_{e,t} p_{e,t}(t + a + b + (n-1)ap_{e,t}),$$

subject to $\sum_{e,t} p_{e,t} = 1$ and $p_{e,t} \geq 0$.

- Optimizing with a Lagrange multiplier we get that the probabilities are identical to the Nash equilibrium of $\ell_e(k) = a_e^* k + b_e^*$ where

$$a_e^* = 2a_e \qquad\qquad b_e^* = b_e - a_e$$

NE and OPT probabilities — Boat model



Price of anarchy — Boat model

- The cost d of each player is

$$d \approx \frac{\sum_e \frac{a_e+b_e}{2(n-1)a_e} + \sqrt{\left(\sum_e \frac{a_e+b_e}{2(n-1)a_e}\right)^2 - \left(\sum_e \frac{1}{2(n-1)a_e}\right)\left(\sum_e \frac{(a_e+b_e)^2}{2(n-1)a_e} - 1\right)}}{\sum_e \frac{1}{2(n-1)a_e}}$$

$$\to \sqrt{\frac{2n}{\sum_e a_e^{-1}}}$$

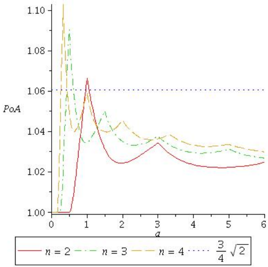
- The optimal cost is

$$d^* = \text{(a similarly complicated expression)}$$
$$\to \frac{4}{3}\sqrt{\frac{n}{\sum_e a_e^{-1}}}$$

- The PoA tends to $3\sqrt{2}/4 \approx 1.06$, as n tends to ∞.

More on the PoA — Boat model

When the number of players is relatively small, the PoA can be higher. Because of the integrality conditions, we analyze only the case of 1 link.



The POA is maximized when

- $a_e = 1/(n-1)$, $b_e = 0$
- Pure equilibrium $p_0 = 1$
- The optimal symmetric solution is $p_0 = 3/4$, $p_1 = 1/4$.
- For these values, we get

$$d = n/(n-1) \quad d^* = (7n+1)/(8(n-1)) \quad PoA = 8n/(7n+1)$$

Remarks — Boat model

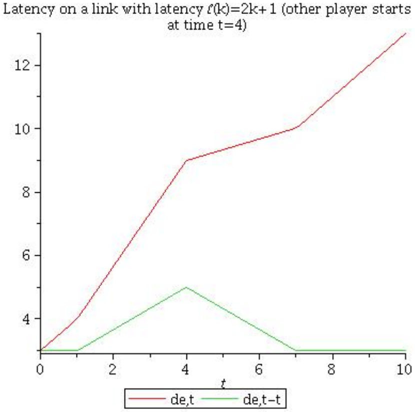
- The optimal solution is the NE of another boat game with latencies $\ell_e(k) = a_e^* k + b_e^*$ where

$$a_e^* = 2a_e \qquad\qquad b_e^* = b_e - a_e$$

- For linear latencies ($b_e = 0$), the parallel links act almost as **parallel resistors** with resistance a_e.
- Strategy can (almost) be partitioned
  - First, select link e with probability proportional to $1/a_e$
  - Then, play the game in link e (with the expected number of players in it)

The structure of the NE — Conveyor belt model

- We consider only 2 players
- $d_{e,t} = t + \ell_e(1) + (\ell_e(2) - \ell_e(1)) \max\left(0, 1 - \frac{|t-t'|}{\ell_e(1)}\right)$



The conveyor belt model for 2 players

NE on one link

- The cost of one player

$$d_t = t + \ell(1) + (\ell(2) - \ell(1)) \sum_{r=-\ell(1)}^{\ell(1)} \left(1 - \frac{|r|}{\ell(1)}\right) p_{t+r}$$

- Crucial step: Show that the support is $\{0, \ldots, h\}$. (But the probabilities are not decreasing!)

- $d_{t+1} - 2d_t + d_{t-1} = p_{t-\ell(1)} - 2p_t + p_{t+\ell(1)}$
- If $t-1$ and $t+1$ are in the support, then t is in the support.
- This argument can be extended to longer intervals

- Putting these together, we find that the probabilities $p_t, p_{t+\ell(1)}, p_{t+2\ell(1)} \ldots$ drop linearly.
- The probabilities are non-zero only at multiples of $\ell(1)$
- With this, it becomes very similar to the boat model

The conveyor belt model for 2 players

Optimal solution in one link

- The situation is similar in the optimal solution: It reduces to the boat model
- In both NE and the optimal symmetric solution:
  - Either the two players do not overlap
  - Or they start together
- As in the boat model. It extends to many links.

Questions (no answers yet)

Open problems

- Conveyor belt model for more players and general networks
- Adaptive strategies: monitor the situation for better timing
- Preemption. Players can abort and start over.

Thank you!