On the Number of Spanning Trees of Multi-Star Related Graphs

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Abstract

In this paper we compute the number of spanning trees of a specific family of graphs using techniques from linear algebra and matrix theory. More specifically, we consider the graphs that result from a complete graph K_n after removing a set of edges that spans a *multi-star* graph $K_m(a_1, a_2, \ldots, a_m)$. We derive closed formulas for the number of spanning trees in the cases of double-star ($m = 2$), triple-star ($m = 3$), and quadruple-star ($m = 4$). Moreover for each case we prove that the graphs with the maximum number of spanning trees are exactly those that result when all the a_i's are equal.

Keywords: Spanning Trees, Multi-Star Graphs, Complement Spanning-Tree Matrix Theorem.

1 Introduction

Let K_n be the complete graph on n vertices and let S be a set of edges that join pairs of vertices in K_n. The problem of calculating the number of spanning trees on K_n that do not contain any edge of S, is a well-known one in graph theory. Many cases have been examined depending on the choice of S. For example, there exist closed formulas for the cases where S is a pairwise disjoint set of edges [9], when it is a star [7], when it is a complete graph [1], when it is a chain of edges [5], and so on (see Berge [1] for an exposition of the main results).

The purpose of this paper is to study the above problem in the cases where S forms *multi-star* graphs (see definition in Section 2). In particular, we derive closed formulas for the cases of double, triple and quadruple stars. Our proofs are based on the *Complement Spanning-Tree Matrix* theorem (CSTM theorem) [8] and use standard linear algebra techniques. Moreover, for each of the three cases, we identify the graphs that possess the maximum number of spanning trees.

The paper is organized as follows. In section 2 we establish the notation and related terminology. In section 3 we present the results obtained for the case of double-stars and the techniques we use for this purpose. In section 4 we show the results for triple and quadruple stars, while section 5 concludes the paper.

2 Preliminaries

The multi-star graph $K_m(a_1, a_2, \ldots, a_m)$ is formed by joining $a_1, a_2, \ldots, a_m$ end-edges to the m nodes of K_m. For example, $K_2(a_1, a_2)$ is the double-star graph [3] and is shown in Figure 1, while the triple $K_3(a_1, a_2, a_3)$ and quadruple $K_4(a_1, a_2, a_3, a_4)$ star graphs, are shown in Figures 2 and 3 respectively.

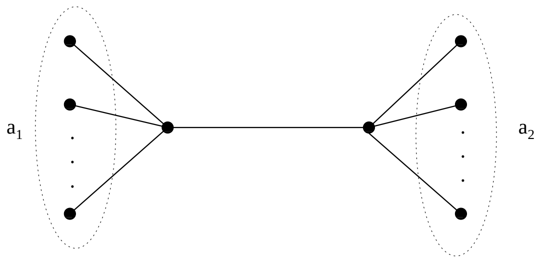

Figure 1: The double-star graph $K_2(a_1, a_2)$.

Given a graph $G = (V, E)$, a subset $S \subseteq E$ of edges *spans* a subgraph $H = (V_S, S)$ where $V_S = \{v \in V \mid v \textit{ is an endpoint of some edge of } S\}$. We consider the family of graphs that results from a complete graph K_n after removing a set of edges that span a multi-star graph $K_m(a_1, a_2, \ldots, a_m)$. Throughout the paper, we refer to this family of graphs as $K_n - K_m(a_1, a_2, \ldots, a_m)$.

Let $G = (V, E)$ be a graph with n vertices and e edges. The *complement* $\bar{G} = (V, \bar{E})$ of G also has V as its vertex set, but two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

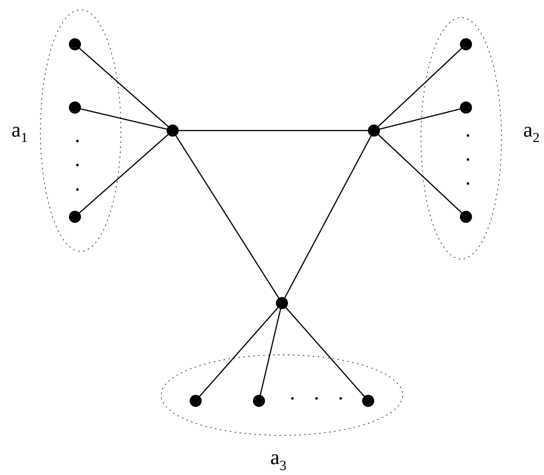

Figure 2: The triple-star graph $K_3(a_1, a_2, a_3)$.

The *Complement Spanning-Tree Matrix* A of a graph G is defined as follows:

$$A(i,j) = \begin{cases} 1 - \frac{d_i}{n} & \text{if } i = j \\ \frac{1}{n} & \text{if } i \neq j,\ (i,j) \in \bar{E} \\ 0 & otherwise \end{cases}$$

where d_i is the number of edges incident to vertex v_i in $\bar{G}$. It has been shown [8], that the number of spanning trees $N(G)$ of G is given by:

$$N(G) = n^{n-2} \cdot Det(A)$$

Given the above definitions, we can now state in a formal way the problem under consideration. We consider the computation of the number of spanning trees of $K_n - K_m(a_1, a_2, \ldots, a_m)$. In particular, we derive closed formulas for the number of spanning trees for the graphs $K_n - K_m(a_1, a_2, \ldots, a_m)$, $m = 2, 3, 4$. Moreover, in each of these cases, we prove that the number of spanning trees is maximized when all a_i's are equal.

3 The Double-Star Case

We use the complement spanning tree matrix theorem in order to compute the number of spanning trees of the graph $K_n - K_2(a_1, a_2)$. We first label the vertices of the graph

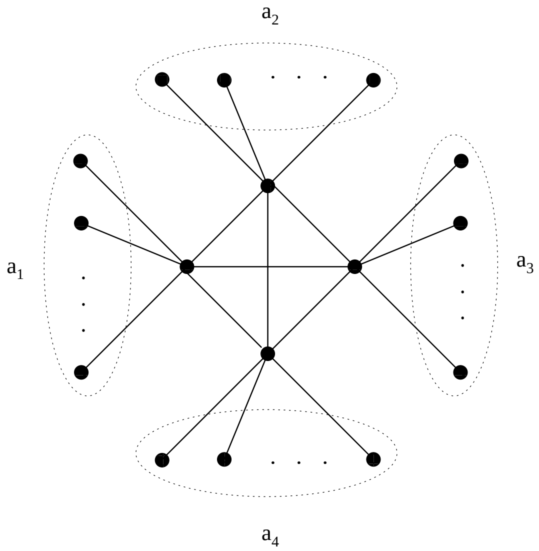

Figure 3: The quadruple-star graph $K_4(a_1, a_2, a_3, a_4)$.

so as that the vertices with degree $n-1$ obtain the smallest labels. We then form the complement spanning tree matrix A of the graph, which has the following form:

$$A = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & 0 & \cdots & 0 \\ & & & 0 & & & \\ & & & \vdots & & B & \\ & & & 0 & & & \end{bmatrix}$$

where the submatrix B concerns those vertices of $K_n - K_2(a_1, a_2)$ that have degree less than $n-1$ (notice that throughout the paper, empty entries in matrices or determinants represent the 0's). Consequently, $Det(A) = Det(B)$, where B is an $(a_1 + a_2 + 2) \times (a_1 + a_2 + 2)$

matrix having the following structure:

$$B = \begin{bmatrix}
a & & & & b & & & & & \\
 & a & & & b & & & & & \\
 & & \ddots & & \vdots & & & & & \\
 & & & a & b & & & & & \\
b & b & \cdots & b & p_1 & & & & & b \\
 & & & & & a & & & & b \\
 & & & & & & a & & & b \\
 & & & & & & & \ddots & & \vdots \\
 & & & & & & & & a & b \\
 & & & & b & b & b & \cdots & b & p_2
\end{bmatrix}$$

where (according to the definition of the Complement Spanning-Tree Matrix) $a = 1 - 1/n$, $b = 1/n$, $p_1 = 1 - (a_1 + 1)/n$ and $p_2 = 1 - (a_2 + 1)/n$. Starting from the upper left part of the matrix, the diagonal has a_1 a's followed by p_1, followed by a_2 a's and ending with p_2.

In order to compute the determinant of the above matrix we start by subtracting row 1 from rows $2, ..., a_1$, and then adding columns $2, \ldots, a_1$ to column 1, getting:

$$Det(B) = \begin{vmatrix}
a & & & & b & & & & & \\
-a & a & & & & & & & & \\
\vdots & & \ddots & & & & & & & \\
-a & & & a & & & & & & \\
b & b & \cdots & b & p_1 & & & & & b \\
 & & & & & a & & & & b \\
 & & & & & & a & & & b \\
 & & & & & & & \ddots & & \vdots \\
 & & & & & & & & a & b \\
 & & & & b & b & b & \cdots & b & p_2
\end{vmatrix} = \begin{vmatrix}
a & & & & b & & & & & \\
 & a & & & & & & & & \\
 & & \ddots & & & & & & & \\
 & & & a & & & & & & \\
a_1 b & b & \cdots & b & p_1 & & & & & b \\
 & & & & & a & & & & b \\
 & & & & & & a & & & b \\
 & & & & & & & \ddots & & \vdots \\
 & & & & & & & & a & b \\
 & & & & b & b & b & \cdots & b & p_2
\end{vmatrix}$$

We multiply the first column by $-(b/a)$ and add it to the (a_1+1) column:

$$\begin{vmatrix} a & & & & & & & & & \\ & a & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & a & & & & & & \\ a_1 b & b & \cdots & b & q_1 & & & & & b \\ & & & & & a & & & & b \\ & & & & & & a & & & b \\ & & & & & & & \ddots & & \vdots \\ & & & & & & & & a & b \\ & & & & b & b & b & \cdots & b & p_2 \end{vmatrix}$$

where $q_1 = p_1 - a_1(b^2/a)$. We now expand, getting :

$$a^{a_1} \begin{vmatrix} q_1 & & & & b \\ & a & & & b \\ & & a & & b \\ & & & \ddots & \vdots \\ b & b & b & \cdots & p_2 \end{vmatrix}$$

By expanding again along row 1, we get:

$$a^{a_1}[q_1 \begin{vmatrix} a & & & b \\ & a & & b \\ & & \ddots & \vdots \\ b & b & \cdots & p_2 \end{vmatrix} + (-1)^{a_2+1} b \begin{vmatrix} 0 & a & & \\ & & \ddots & \\ & & & a \\ b & b & \cdots & b \end{vmatrix}]$$

The second determinant is easily shown to be equal to $(-1)^{a_2} b a^{a_2}$, and therefore the determinant of matrix B becomes:

$$Det(B) = a^{a_1}[q_1 \begin{vmatrix} a & & & b \\ & \ddots & & \vdots \\ & & a & b \\ b & \cdots & b & p_2 \end{vmatrix} - b^2 a^{a_2}] \tag{1}$$

It now suffices to compute the remaining determinant. We follow a procedure similar to the one we applied for the initial matrix. We start by subtracting row 1 from rows

$2, \ldots, a_2$, and then adding columns $2, \ldots, a_2$ to column 1, getting:

$$\begin{vmatrix} a & & & & b \\ -a & a & & & \\ \vdots & & \ddots & & \\ -a & & & a & \\ b & b & \cdots & b & p_2 \end{vmatrix} = \begin{vmatrix} a & & & & b \\ & a & & & \\ & & \ddots & & \\ & & & a & \\ a_2 b & b & \cdots & b & p_2 \end{vmatrix}$$

We multiply the first column by $-(b/a)$ and add it to the (a_2+1) column:

$$\begin{vmatrix} a & & & & \\ & a & & & \\ & & \ddots & & \\ & & & a & \\ a_2 b & b & \cdots & b & q_2 \end{vmatrix}$$

where $q_2 = p_2 - a_2(b^2/a)$. As the above matrix is a lower triangular, the value of the determinant is equal to $a^{a_2}q_2$. Substituting this into equation (1), we get the value of the determinant $Det(A)$ of the initial matrix:

$$Det(A) = a^{a_1}[q_1 a^{a_2} q_2 - b^2 a^{a_2}] = a^{a_1+a_2}[q_1 q_2 - b^2]$$

Based on the formula that gives the number $N(G)$ of spanning trees of a graph G, we have the following Theorem:

Theorem 3.1 The number of spanning trees of the graph $G = K_n - K_2(a_1, a_2)$ is

$$N(G) = n^{n-2} a^{a_1+a_2}[q_1 q_2 - b^2]$$

where $a = 1 - 1/n$, $b = 1/n$, $p_i = 1 - (a_i + 1)/n$, and $q_i = p_i - a_i(b^2/a)$, $i = 1, 2$.

It is clear from the above theorem that the number of spanning trees of the graph $K_n - K_2(a_1, a_2)$ depends on the values of a_1 and a_2. We are interested in determining the particular graph which has the maximum number of spanning trees. Therefore, we simply need to find the values of a_1 and a_2 that maximize the formula of Theorem 3.1.

Theorem 3.2 The number of spanning trees of the graph $K_n - K_2(a_1, a_2)$ is maximized when $a_1 = a_2$.

Proof: It is clear that in order to maximize the value of the formula in Theorem 3.1 we must maximize the quantity $q_1 q_2$. It is straightforward to see that $q_1 + q_2$ is constant, and therefore the value of $q_1 q_2$ is maximized when $q_1 = q_2$. This easily leads to the fact that $a_1 = a_2$ must hold in order for the maximization to be obtained. ■

4 The Triple and Quadruple-Star Cases

In this section we consider the cases of the triple and quadruple star graphs. We omit the details of the calculations as they are based on the same principles as the ones for the double-star case.

We consider first the case of the $K_n - K_3(a_1, a_2, a_3)$ graph. The matrix B that results for this graph (based on the Complement Spanning-Tree Matrix theorem) has the following form:

$$B = \left[\begin{array}{cccccccccccc}
a & & & b & & & & & & & & \\
& \ddots & & \vdots & & & & & & & & \\
& & a & b & & & & & & & & \\
b & \cdots & b & p_1 & & & & b & & & & b \\
& & & & a & & & b & & & & \\
& & & & & \ddots & & \vdots & & & & \\
& & & & & & a & b & & & & \\
& & & b & b & \cdots & b & p_2 & & & & b \\
& & & & & & & & a & & & b \\
& & & & & & & & & \ddots & & \vdots \\
& & & & & & & & & & a & b \\
& & & b & & & & b & b & \cdots & b & p_3
\end{array}\right]$$

where $a = 1 - 1/n$, $b = 1/n$, $p_1 = 1 - (a_1 + 2)/n$, $p_2 = 1 - (a_2 + 2)/n$ and $p_3 = 1 - (a_3 + 2)/n$.

Although the structure of the above matrix is similar to the matrix of the double-star case, it is not obvious how to reduce the determinant of the triple-star case to a form that will be immediately calculable with a procedure identical to the one of the double-star. In other words, the calculation for the triple-star case has to be performed from scratch (using however similar techniques).

It can be shown (we avoid the technical details) that the following theorem holds.

Theorem 4.1 The number of spanning trees of the graph $G = K_n - K_3(a_1, a_2, a_3)$ is

$$N(G) = n^{n-2} a^{a_1+a_2+a_3} [q_1 q_2 q_3 - b^2(q_1 + q_2 + q_3) + 2b^3]$$

where $a = 1 - 1/n$, $b = 1/n$, $p_i = 1 - (a_i + 2)/n$, and $q_i = p_i - a_i(b^2/a)$, $i = 1, 2, 3$.

We consider now the case of the $K_n - K_4(a_1, a_2, a_3, a_4)$ graph. Again, there is a close relationship between the matrices of the quadruple and triple star cases, but still this relationship is not strong enough so as to ensure a straightforward calculation. It can be shown (using similar techniques) that the following theorem holds.

Theorem 4.2 The number of spanning trees of the graph $G = K_n - K_4(a_1, a_2, a_3, a_4)$ is

$$\begin{aligned} N(G) &= n^{n-2} a^{a_1+a_2+a_3+a_4} [q_1 q_2 q_3 q_4 - \\ &\quad b^2(q_1 q_2 + q_1 q_3 + q_1 q_4 + q_2 q_3 + q_2 q_4 + q_3 q_4) + 2b^3(q_1 + q_2 + q_3 + q_4) - 3b^4] \end{aligned}$$

where $a = 1 - 1/n$, $b = 1/n$, $p_i = 1 - (a_i + 3)/n$, and $q_i = p_i - a_i(b^2/a)$, $i = 1, 2, 3, 4$.

A similar maximization theorem as in the double-star case can be proved for the triple and quadruple cases. Thus, we can state the following result.

Theorem 4.3 The number of spanning trees of the graphs $K_n - K_3(a_1, a_2, a_3)$ and $K_n - K_4(a_1, a_2, a_3, a_4)$ is maximized when the a_i's are equal.

Proof: Using a similar (but slightly more involved) technique than the case of double-star graphs. ■

5 Discussion

In this paper, a number of closed formulas regarding the number of spanning trees of multi-star related graphs have been derived. For this purpose we have used the Complement Spanning Tree Matrix Theorem as well as standard techniques from linear algebra and matrix theory. For each case, we have determined the particular multi-star graphs that maximize the number of spanning trees.

Calculating the determinant of the Complement Spanning Tree Matrix seems to be a promising approach for computing the number of spanning trees of families of graphs of

Graph	a_1	a_2	a_3	a_4	Known Results	Reference
$K_n - K_2(a_1, a_2)$	a_1	0			$K_n - K_{1,a_1}$	O'Neil [7]
$K_n - K_2(a_1, a_2)$	0	0			$K_n - P_2$	Temperley [8] (also [5])
$K_n - K_2(a_1, a_2)$	0	1			$K_n - P_3$	Moon [5] (also [7])
$K_n - K_2(a_1, a_2)$	1	1			$K_n - P_4$	Moon [5]
$K_n - K_3(a_1, a_2, a_3)$	0	0	0		$K_n - K_3$	O'Neil [7]
$K_n - K_4(a_1, a_2, a_3, a_4)$	0	0	0	0	$K_n - K_4$	O'Neil [7]

Table 1: Results obtained as special cases of multi-star graphs.

the form $K_n - G$, where G possess an inherent symmetry. In particular, many of the well-known results in Berge [1] which are derived using combinatorial arguments, can easily be proved using similar techniques to the ones we have used in this paper. More specifically, many graphs can be derived as special cases from the multi-star graphs, depending on the values of the a_i's. For example, given a double-star $K_2(a_1, a_2)$ and setting $a_2 = 0$, we get the *star* on $a_1 + 1$ vertices, and when setting $a_1 = a_2 = 0$ we get the *path* graph on two vertices P_2. A listing of such results is presented in Table 1.

Deriving closed formulas for different types of graphs can prove to be helpful in identifying those graphs that contain the maximum number of spanning trees. Such an investigation has practical consequences related to network reliability (see for example [2, 4, 6]).

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